ALAGAPPA UNIVERSITY

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KARAIKUDI – 630 003

Directorate of Distance Education

B.Sc. (Mathematics)

I - Semester

113 14

CALCULUS

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Units (1-14)

SYLLABI-BOOK MAPPING TABLE

Calculus

Syllabi	Mapping in Book

BLOCK I: DIFFERENTIATION, POLAR CO-ORDINATES AND ASYMPTOTES

UNIT-I: Differentiation – Introduction, Parametric differentiation, Logarithmic differentiation, differentiation of implicit functions.

UNIT-II: Successive differentiation – Introduction, nth derivative of some standard functions, problems using higher order derivatives.

UNIT-III: Partial differentiation – Homogeneous functions, Euler's theorem, verification of Euler's theorem, Maxima and minima of functions of one variable and two variables.

UNIT-IV: Polar Coordinates – Radius of curvature in polar coordinates, p-r equation of a curve – Asymptotes – Method of finding asymptotes – problems

Unit 1: Differentiation
(Pages 1-23);
Unit 2: Successive Differentiation and Standard Functions
(Pages 24-39);
Unit 3: Partial Differentiation
(Pages 40-66);
Unit 4: Polar Coordinates and Asymptotes
(Pages 67-84)

BLOCK II: ENVELOPES, EVOLUTES AND INTEGRALS

UNIT-V: Tangents and normal angle of intersection, curvature, Envelopes and Evolutes, working method to find envelope and involutes.

UNIT-VI: Integration – Substitution methods, $1/(x_2 - a_2)$, $1/(x_2 + a_2)$, $1/(a_2 - x_2)$, $1/(x_2 - a_2)^{1/2}$, $(x_2 - a_2)^{1/2}$, $(x_2 + a_2)^{1/2}$, $(a_2 - x_2)^{1/2}$.

UNIT-VII: Definite Integrals and their properties problems – Integration by parts – Reduction formulae – Bernoulli's formula.

UNIT-VIII: Double and triple integrals and their properties – Jacobian Change of order of integration.

Unit 5: Tangents, Curvature, Envelopes and Evolutes
(Pages 85-102);
Unit 6: Integration
(Pages 103-116);
Unit 7: Integration – Definite Integrals
(Pages 117-135);
Unit 8: Double and Triple Integrals
(Pages 136-162)

BLOCK III: BETA, GAMMA FUNCTIONS AND SOLUTIONS OF DIFFERENTIAL EQUATIONS

UNIT-IX: Beta and Gamma functions – properties – problems

UNIT-X: Differential equations – Solution of differential equations, variable separable methods.

UNIT-XI: Homogeneous equations in x and y-Methods and problems, First order linear equations.

Unit 9: Beta and Gamma Functions
(Pages 163-181);
Unit 10: Differential Equations
(Pages 182-198);
Unit 11: Homogeneous Equations and First Order Linear Equations
(Pages 199-212)

BLOCK IV: VARIATIONS OF PARAMETERS, LAPLACE TRANSFORMATIONS AND STANDARD FORMS OF PARTIAL DIFFERENTIAL EQUATIONS

UNIT-XII: Linear equations of order 2 with constant and variable coefficients, Variation of parameters.

UNIT-XIII: Laplace transform, Inverse Laplace transform, Solving differential equations using Laplace transforms.

UNIT-XIV: Partial differential equations – Forming differential equations by eliminating arbitrary constants and variables, First order partial order equations. Some standard forms – Charpit's method, Clairaut's form, Lagrange's multiplier method and problems.

Unit 12: Linear Equations of Order 2 and Variation of Parameters
(Pages 213-222);
Unit 13: Laplace Transform
(Pages 223-234);
Unit 14: Partial Differential Equations
(Pages 235-251)

CONTENTS

INTRODUCTION

Calculus is the mathematical study of continuous change. The discovery of calculus is often attributed to two men, Isaac Newton and Gottfried Leibniz, who independently developed its foundations. It has two major branches differential calculus and integral calculus. Differential calculus is concerned with instantaneous rates of change and slopes of curves, and integral calculus is concerned with accumulation of quantities and the areas under and between curves.

One should understand that the main purpose of learning calculus is not just knowing about differentiation and integration but also knowing how to apply differentiation and integration to solve problems. For that, one must have sound understanding of the concepts. If you are a beginner in calculus, then be sure that you have had the appropriate knowledge of algebra and trigonometry.

This book, *Calculus*, is divided into four blocks, which are further subdivided into fourteen units. The first unit introduces the concept of differentiation while successive differentiation has been discussed in the following unit. The third unit deals with the concept of partial differentiation and Euler's theorem. Polar coordinates and asymptotes are focused on in the fourth unit while the fifth unit explains the concept of envelopes and evolutes. Integration by substitution method has been discussed in the sixth unit while seventh unit deals with definite integrals and integration by parts. Eighth units discusses the concept of double and triple integrals and their properties and Beta and Gamma functions have been explained in the following unit. Differential equations have been explained in the tenth unit while eleventh units focuses on homogeneous equations and first order linear equations. Twelfth unit discusses about linear equations of order two while thirteenth unit introduces you to the concept of Laplace transform. The last units discusses partial differential equations and various method to form them.

The book follows the self-instructional mode wherein each unit begins with an Introduction to the topic. The Objectives are then outlined before going on to the presentation of the detailed content in a simple and structured format. Check Your Progress questions are provided at regular intervals to test the student's understanding of the subject. A Summary, a list of Key Words and a set of Self-Assessment Questions and Exercises are provided at the end of each unit for recapitulation.

BLOCK - I
DIFFERENTIATION, POLAR CO-ORDINATES
AND ASYMPTOTES

UNIT 1   DIFFERENTIATION

Structure

1.0   INTRODUCTION

This unit introduces you to the concept of differentiation. Differentiation plays an important part in mathematics. Differentiation has applications to nearly all quantitative disciplines. For example, in physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of velocity with respect to time is acceleration. The derivative of the momentum of a body equals the force applied to the body; rearranging this derivative statement leads to the famous F = ma equation associated with Newton's second law of motion. After defining differentiation, this unit takes you to the different methods used to differentiate a function. There are two methods discussed here - parametric differentiation and logarithmic differentiation. Both of these methods have their own unique importance in calculus. In the end, you will learn the differentiation of implicit functions.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of differentiation
- Learn the process of parametric differentiation

• Learn the process of logarithmic differentiation

• Discuss differentiation of implicit functions

1.2 DIFFERENTIATION – AN INTRODUCTION

Differentiation method is specifically used for finding or estimating the **rate of change** when one quantity is compared with another, precisely when the rate of change is not constant. Following are some definitions of differentiation.

Definitions

1. In mathematics, '**Differentiation**' is the process of finding the **derivative**, or **rate of change**, of a **function**.

2. **Differentiation** is the process of finding or evaluating the **rate of change** of **dependent quantity** with respect to **independent quantity**, i.e., $y = fx$ where y is **dependent** with respect to x.

As defined above, differentiation method is used for "*finding a function that outputs the rate of change of one variable with respect to another variable*".

In **differential calculus**, the key objects are the **derivative of a function** and interrelated notions, such as the differential and its various applications. Fundamentally, the derivative of a function at a selected input value defines the rate of change of the function that is close or adjacent to that particular input value. Thus, **differentiation** is the method used to find a **derivative**. Geometrically, the derivative of a function at a specified point is approximated as the slope of the tangent line on the graph of the function at the specified point, as long as the derivative exists and is well-defined at that selected point. **Derivatives** are often used for finding the **maxima** and **minima** of a function and the equations which involve the derivatives are termed as **differential equations**.

Normally, when we consider a real-valued function of a **single real variable**, then the **derivative of a function** at specified or selected point usually defines or determines the **best linear approximation** to the **function** at that specified or selected point. The derivative is used for determining the maximum and minimum values of specific or precise functions, such as profit, loss, cost, etc.

Both the **calculus**, the **differential** and the **integral** are linked or interrelated through the fundamental theorem of calculus, which specifies that the differentiation method is the reverse or opposite to integration method.

Differentiation has its applications in almost all types of quantitative disciplines and helps in solving various types of real-world problems, such as in the field of physics, the displacement derivative of any object in motion with respect to time is termed as the velocity of the moving object, and this derivative of velocity with respect to time is termed as acceleration. Furthermore, in chemistry the derivative can also be approximated for the reaction rate of any chemical reaction while in

the operations research, the derivatives are specifically used for efficiently determining the methods for transporting materials or supplies and designing factories.

In various fields of mathematics, the derivatives and their generalizations are frequently used for solving the problems related to functional analysis, complex analysis, differential geometry, abstract algebra and measure theory.

Let us understand the concept of differentiation with the help of a simple example. Assume that we have to track the position of a moving vehicle on a road. To track its position we assign it a variable 'x'. In the meantime the position of the moving vehicle will change as the time changes, i.e., the variable x is dependent on time or in other words $x = f(t)$. Differentiation provides a function dx/dt which characterizes or represents the speed of the moving vehicle, i.e., the rate of change of the position of moving vehicle with respect to time.

Approximating the 'Rate of Change' which is 'Not Constant'

When we throw a round disc straight up in the air, then after some time the speed of disc slows down as the gravity or the gravitational force acts on the disc. As a result the disc direction changes and it starts falling down. During this whole motion, i.e., when the disc was moving straight up and when the disc was falling down the velocity keeps changing continuously. Basically, the motion gradually changes from positive (when the disc is thrown straight up) to negative, i.e., the motion decreases and becomes zero (when the disc is falling downwards). Therefore, when the disc is moving straight up then it has negative acceleration while the acceleration becomes positive when the disc is falling down.

Figure 1.1 shows the graph of disc height (in metres) against time (in seconds).
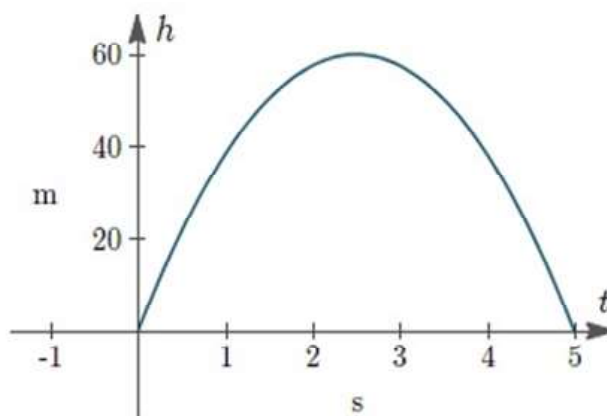
Fig. 1.1 *Graph of Disc Height at Time 't'*

In the Figure 1.1 the slope of the disc height at time t graph is continuously changing or varying while in motion. Initially or at the starting, the graph shows a steep positive slope which indicates that the velocity is large or enormous at the time when we throw the disc straight up. Subsequently, the slope decreases

gradually or becomes less until it reaches 0, i.e., when the disc is at its maximum point then the velocity becomes zero (refer Figure 1.2). When the disc starts falling downwards then the slope is referred as negative since it corresponds to the negative velocity as shown in Figure 1.2.
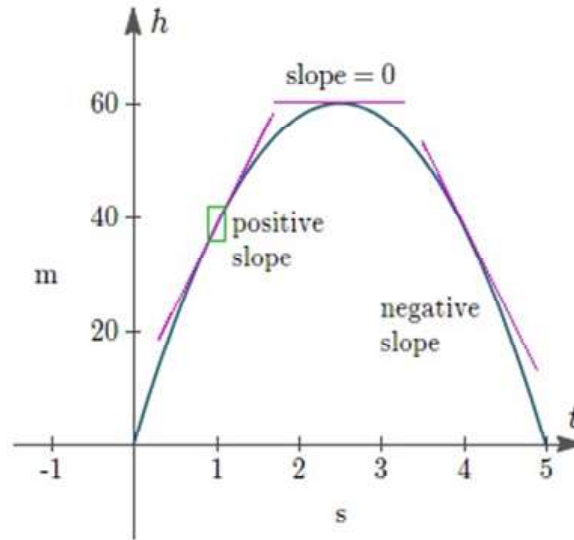
Fig. 1.2 *Graph of Disc Height Showing Slopes*

Now if we zoom in the graph at the section near $t = 1$, i.e., the rectangular section marked in Figure 1.2, then it looks as shown in Figure 1.3, zoomed view. The approximation is calculated on the section between $t = 0.9$ s and $t = 1.1$ s.
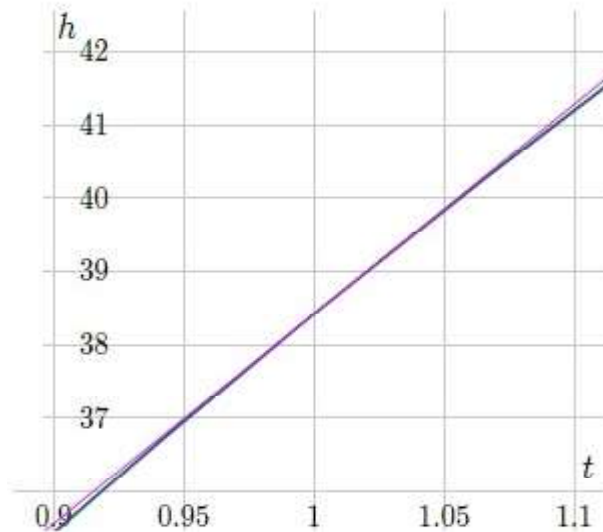
Fig. 1.3 *Zoomed Section of Graph between t = 0.9 s and t = 1.1 s*

When the selected curved section of graph is zoomed in then the curved line appears to be as a straight line. Almost perfect approximation can be made

about the slope of the curve at the specified point $t = 1$, which is referred as the slope of the tangent to the curve (see the dim line near the straight line). The approximation is made by identifying or observing all those points through which the curve passes near $t = 1$. Mathematically, a tangent is defined as specific line which touches the curve only at one point.

Perceiving the graph, we comprehend that it passes through the points (0.9, 36.2) and (1.1, 42). Therefore the slope of the tangent at $t = 1$ is approximated as follows:

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{42.0 - 36.2}{1.1 - 0.9}$$

$$= \frac{5.8}{0.2}$$

$$= 29 \text{ m/s}$$

Since this is velocity therefore the unit of measurement is **m/s**. The rate of change is estimated by observing the slope.

Derivative

Assume that for the real numbers x and y, '**y**' is a **function** of '**x**', i.e., for each and every value of x, there will always be a corresponding value of y. We can write this equation of x and y relationship as,

$y = f(x)$

If $f(x)$ is considered as the equation, also termed as **linear equation**, for a straight line, then there exist two real numbers m and b such that $y = mx + b$. In the slope intercept method, the 'm' defines the slope and is determined or evaluated using the following formula:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

Here the symbol Δ (Greek letter Delta) is an abbreviation that represents '**change in**'. Therefore $\Delta y = m \, \Delta x$.
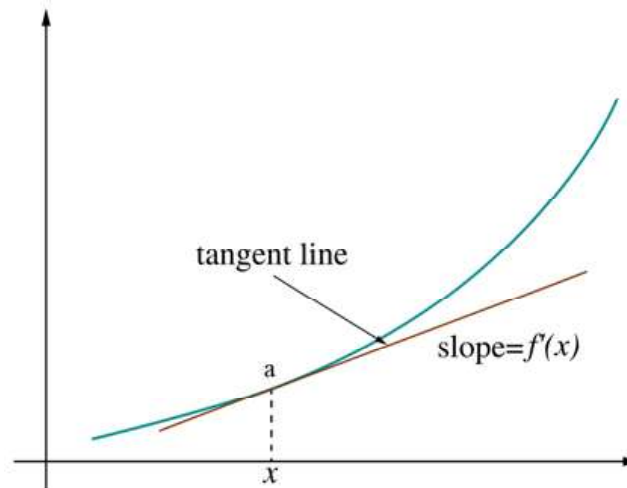
Fig. 1.4 *The Tangent Line at (x, f(x))*

Since the general or common function has no line, hence it has no slope. As per geometric rules, "the **derivative** of *f* at the **point** *x* = *a* is the **slope** of the **tangent line** of the **function** *f* at the **specified point** *a*" as shown in Figure 1.4.

This is frequently denoted or represented as *f′* (*a*) in Lagrange's notation and as in $\frac{dy}{dx}\big|_{x=a}$ Leibniz's notation. Because the derivative is defined as the slope of the linear approximation towards *f* at the specified point *a*, therefore the derivative in conjunction with the value of *f* at *a* basically determines or approximates the best 'linear approximation' or the 'linearization' of function *f* near the specified point '*a*'.

In the domain of '*f*' there is derivative for each point '*a*' which defines a function for sending every specified point '*a*' to the derivative of '*f*' at '*a*', such as when *f*(*x*) = *x*², then at that point the derivative function is $f'(x) = \frac{dy}{dx} = 2x$.

Another related significant notion is the '***differential of a function***'. For the real variables *x* and *y*, we can state that the derivative of *f* at *x* is the slope of the tangent line for the required graph of *f* at *x*. Additionally, the derivative of '*f*' will be real number because both the source and the target of function *f* are one-dimensional. Now if we take the best linear approximation in a single direction then we can determine a partial derivative denoted as ∂*y*/∂*x*. The total derivative can be defined as the linearization of *f* in all directions simultaneously.

Check Your Progress

1. What is differentiation?
2. What is Lagrange's notation of derivative of a function *f*(*x*)?

1.3   PARAMETRIC DIFFERENTIATION

As already discussed in the preceding section that to find or obtain the derivative of an expression where one variable is the dependent variable generally named 'y' is typically expressed as a function of another independent variable generally named 'x', mathematically written as $y = f(x)$.

However this is not always be the situation. Occasionally we come across characteristic situations where we can not possibly express y in terms of x and vice versa. Alternatively we can express both the variables x and y in terms of a third variable named 't' by convention possibly because this variable is frequently used for representing time. This third variable is termed as parameter.

A function that contains this **third variable** or **parameter** is termed as **parametric function** and the method of **differentiating** a **parametric function** is defined as the **parametric differentiation**. Though, to differentiate a parametric function is slightly more difficult as comparison to differentiate a function which has only two variables.

A **parametric derivative**, in calculus, is approximated using a derivative of a dependent variable 'y' with reference to an independent variable 'x' specifically taken in the condition when both the variables typically depend on an third independent variable 't', normally assumed as '**time**', i.e., specifically when the variables x and y are defined through the parametric equations in t.

First Derivative

Consider that $x(t)$ and $y(t)$ are the coordinates of the points of the curve that is expressed or stated as follows by means of functions of a variable 't':

$$y = y(t), \ x = x(t)$$

Therefore the first derivative is precisely implied using these parametric equations is given as,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}(t)}{\dot{x}(t)}.$$

Here the notation $\dot{x}(t)$ denotes or represents the derivative of x with respect to t. We can also derive the equation by means of the **chain rule** for **derivatives** as follows:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Now when we divide both sides by dx/dt we obtain the equation derived above.

As a general rule, all of the derivatives dy/dt, dx/dt and dy/dx are themselves considered as functions of t and therefore can more explicitly be written as,

$$\frac{dy}{dx}(t)$$

Second Derivative

In parametric differentiation, the second derivative that is precisely implied using a parametric equation is given by the following equations:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{1}{\dot{x}}$$

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

This is approximated using the **quotient rule** for **derivatives**.

Example 1. Let us take the set of functions in which $x(t) = 4t^2$ and $y(t) = 3t$.

Differentiating both functions with regard to t gives the equations,

$$\frac{dx}{dt} = 8t$$

And,

$$\frac{dy}{dt} = 3 \text{, respectively.}$$

The following equation is derived when we substitute the above into the formula for the parametric derivative,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3}{8t}.$$

Here $\dot{x}$ and $\dot{y}$ are assumed to be functions of t.

Differentiating parametric equations using the **chain rule**.

Example 2. If $x = 2at^2$ and $y = 4at$, then find dy/dx.

Solution: We know that, $dy/dx = dy/dt \times dt/dx$

Here,

$dx/dt = 4at$ and therefore $dt/dx = 1/(4at)$

Also,

$$dy/dt = 4a$$

Hence,

$$dy/dx = 4a \times 1/4at = 1/t$$

Finding the Second Derivative

To find the second derivative we proceed using the following equation in the Example 2.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

Therefore, we can obtain the second derivative as follows:

$$d^2y / dx^2 = d(1/t) / dt \times dt / dx$$

$$= -1 / t^2 \times 1 / 4at$$

Derivatives of Parametric Functions

In parametric form, the relationship between the variables x and y is expressed with the help of following two equations:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Here the variable t is termed as **parameter**.

For instance, following two functions define the equation of a circle that is centered at the origin with the radius R, in parametric form. In this situation, the parameter t fluctuates or varies from 0 to 2π.

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$$

To find or obtain an expression for the derivative of a parametrically defined function let us assume that the functions $x = x(t)$ and $y = y(t)$ can be differentiated in the interval $\alpha < t < \beta$ and $x'(t) \neq 0$. Additionally, we can also assume that the function $x = x(t)$ has an inverse function $t = \varphi(x)$.

According to the **inverse function theorem**,

$$\frac{dt}{dx} = t'_x = \frac{1}{x'_t}$$

We can consider the original function $y(x)$ as a composite function of the form,

$$y(x) = y(t(x))$$

Its derivative is now given as follows,

$$y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}$$

Using the above formula we can find or obtain the derivative of a parametrically defined function where there is no need to express the function $y(x)$ in explicit form.

Following examples will explain that how we can find or approximate the derivative of the parametric function.

Example 3. Find the derivative of the parametric function when $x = t^2$ and $y = t^3$.

Solution: For finding the derivatives of x and y we use the third variable t or we can find with reference to t as follows:

$$x'_t = \left(t^2\right)' = 2t, \ y'_t = \left(t^3\right)' = 3t^2.$$

Therefore,

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3t^2}{2t} = \frac{3t}{2} \ (t \neq 0)$$

Example 4. Find the derivative of the parametric function when $x = 2t + 1$ and $y = 4t - 3$.

Solution: Similarly as above for finding the derivatives of x and y we use the third variable t or we can find with reference to t as follows:

$$x'_t = (2t + 1)' = 2, \ y'_t = (4t - 3)' = 4.$$

Subsequently,

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{4}{2} = 2.$$

Example 5. Find the derivative of the parametric function when $x = e^{2t}$ and $y = e^{3t}$.

Solution: To find the derivatives of x and y we use the equation of the form,

$$x'_t = \left(e^{2t}\right)' = 2e^{2t}, \ y'_t = \left(e^{3t}\right)' = 3e^{3t}.$$

Therefore, the derivative dy/dx is specified by the following:

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{3t-2t} = \frac{3}{2}e^t$$

Example 6. Find the derivative of the parametric function when $x = 2t^2 + t + 1$ and $y = 8t^3 + 3t^2 + 2$.

Solution: For finding the derivative of the parametric function we differentiate both the equations with respect to the parameter t as follows:

$$x'_t = \left(2t^2 + t + 1\right)' = 4t + 1, \ \ y'_t = \left(8t^3 + 3t^2 + 2\right)' = 24t^2 + 6t.$$

Therefore, the derivative dy/dx is specified by the following:

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{24t^2 + 6t}{4t + 1} = \frac{6t\,(4t+1)}{4t+1} = 6t.$$

Example 7. Find the derivative of the parametric function when $x = a \cos t$ and $y = b \sin t$.

Solution: The equations define that an ellipse is centered specifically at the origin through semiaxes a and b.

For finding the derivative of the parametric function we differentiate the variables x and y with regard to the parameter t as follows:

$$x'_t = (a \cos t)' = -a \sin t, \ \ y'_t = (b \sin t)' = b \cos t.$$

Consequently, the derivative dy/dx is dependent on the parameter t as shown below in the equations:

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{b \cos t}{(-a \sin t)} = -\frac{b}{a}\cot t.$$

In this state, the parameter t varies from $-\pi$ to π. Though the derivative dy/dx is infinite at the points $t = 0, \pm\pi$.

Consequently, this domain is represented as $0 < |t| < \pi$.

Example 8. Given are the following parametric equations:

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for} \quad 0 \le t \le 2\pi$$

Plot a graph for this.

Note: Here both x and y are specified in terms of the third variable t.

Solution: To plot a graph we will first plot graphs of $\cos t$ and $\sin t$ as shown below in the given Figure 1.
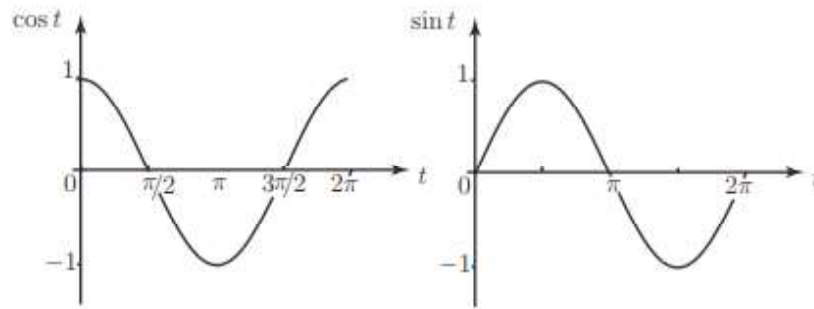
Fig. 1.5 *Graphs of cos t and sin t*

Evidently,

For when $t=0, x=\cos 0=1; y=\sin 0=0.$ (1)

And when $t=\pi/2, x=\cos \pi/2=0; y=\sin \pi/2=1$.   (2)

By this method we find or get the x and y coordinates of large numbers of points specified by Equations 1 and 2. Refer Table 1 for the values of x and y as specified by Equations 1 and 2.

Table 1. Values of x and y as Specified by Equation 1

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

To obtain a graph, we plot the points specified by the x and y coordinates as given in Table 1. Now we join these points by drawing a smooth curve to get the perfect graph as shown below in Figure 1.6. The parametric equations specify a circle which is centered at the origin and is of Radius 1.
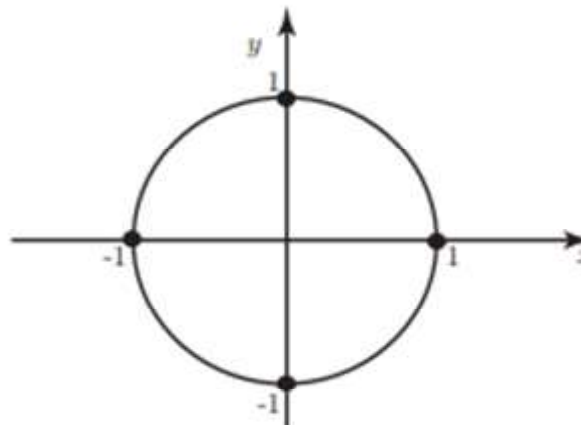
Fig. 1.6 *Circle Centered at the Origin with Radius 1*

Therefore, $x = \cos t$, $y = \sin t$, for t positioned between 0 and 2π, are termed as the parametric equations that define a circle having centre (0, 0) and Radius 1.

Point to Remember

Parametric Differentiation

If,

$$x = x(t) \text{ and } y = y(t)$$

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Provided that, $\dfrac{\mathrm{d}x}{\mathrm{d}t} \neq 0$

Example 9. Find dy/dx for $x = \cos t$ and $y = \sin t$.

Solution: For finding dy/dx, differentiate both x and y with regard to the parameter 't' as follows:

$$dx/dt = -\sin t \qquad dy/dt = \cos t$$

According to the chain rule we get the following equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Consequently, by rearranging we obtain,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Provided that dx/dt is not equal to '0'.
Therefore, in this condition,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\cos t}{-\sin t} = -\cot t$$

Check Your Progress

3. Define parametric differentiation.
4. If $x = 3t$ and $y = 4t$, then find dy/dx.

1.4 LOGARITHMIC DIFFERENTIATION

In calculus, **logarithmic differentiation** or **differentiation by using logarithms** is a specific method to differentiate functions by means of the **logarithmic derivative** of a function 'f'. This method is used in when it is easy to differentiate the logarithm of a function instead of the function itself.

In addition to the properties of logarithms, specifically the **natural logarithm** or the logarithm to the **base e**, '**Logarithmic Differentiation**' depends on the **chain rule** for transforming the products into sums and divisions into subtractions.

The logarithmic principle is applied in the differentiation of all differentiable functions provided that the functions are non-zero.

For a function,

$$y = f(x)$$

Typically the logarithmic differentiation takes place on both sides with the natural logarithm or the logarithm to the base **e**, taking the absolute values as follows:

$$\ln |y| = \ln |f(x)|.$$

After implicit differentiation we have the following form:

$$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

On the left-hand side **multiply by y** so that **1/y** is eliminated and only **dy/dx** is left behind.

$$\frac{dy}{dx} = y \times \frac{f'(x)}{f(x)} = f'(x).$$

This method is very helpful in differentiation because the properties of logarithms provide possibilities for quickly simplifying complex functions. Following are the utmost and frequently used logarithm laws:

$$\ln(ab) = \ln(a) + \ln(b), \qquad \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b), \qquad \ln(a^n) = n\ln(a).$$

Logarithmic differentiation thus provides method for differentiating a function. However, for logarithm of any base the following given properties and methods are accurate and true, but we will be using only the natural logarithm ($\ln$), i.e., base e where e ≈ 2.718281828.

Therefore, the logarithmic differentiation is the method or process of differentiating functions by first obtaining or finding the logarithms and then differentiating it. Using this method you can efficiently calculate the derivatives of power, rational functions and also some irrational functions.

Let us understand the concept of logarithmic differentiation with the help of an example.

Let, $y = f(x)$

Now taking the natural logarithms of both sides of the equation, we obtain:

$\ln y = \ln f(x)$

Subsequently, we apply the chain rule to differentiate the above expression since y is a function of x.

$$(\ln y)' = (\ln f(x))', \quad \Rightarrow \frac{1}{y} y'(x) = (\ln f(x))'.$$

It is comprehended that the derivative is,

$$y' = y(\ln f(x))' = f(x)(\ln f(x))'.$$

Fundamentally, the **derivative** of the **logarithmic function** is termed as the **logarithmic derivative** of the **initial function** $y = f(x)$.

Using the logarithmic differentiation method we can also efficiently compute the **derivatives** of **power-exponential functions**, i.e., functions of the form,

$$y = u(x)^{v(x)}$$

Here $u(x)$ and $v(x)$ are the differentiable functions of x.

Properties of Natural Logarithm

1. $\ln 1 = 0$.
2. $\ln e = 1$.
3. $\ln e^x = x$.
4. $\ln y^x = x \ln y$.
5. $\ln(xy) = \ln x + \ln y$.
6. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$.

Following are some of the common mistakes which should be avoided.

1. $\ln(x + y) = \ln x + \ln y$.
2. $\ln(x - y) = \ln x - \ln y$.
3. $\ln(xy) = \ln x \ \ln y$.

4. $\ln\left(\dfrac{x}{y}\right) = \dfrac{\ln x}{\ln y}$.

5. $\dfrac{\ln x}{\ln y} = \ln x - \ln y$.

The following examples will help you to understand the use of logarithmic differentiation for obtaining the derivative of the function $y(x)$.

Example 10. Differentiate $y = x^x$.

Solution: In this function, since the variable is raised to a variable power therefore the differentiation can not be done using the ordinary rules of differentiation. To obtain a derivative of $y = x^x$, on both sides of this equation we apply the natural logarithm as follows.

$$\ln y = \ln x^x$$

$$= x \ln x.$$

Now differentiate both the sides of the obtained equation. For differentiation on the left-hand side is done using the chain rule because here y represents a function of x while the differentiation on the right-hand side is done using the product rule. Thus, to differentiate we start with the equation,

$$\ln y = x \ln x$$

On differentiating, we obtain the following equation:

$$\frac{1}{y}y' = x\frac{1}{x} + (1)\ln x$$

$$= 1 + \ln x$$

On multiplying both the sides of the above equation by y, we obtain:

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

Example 11. Differentiate $y = x^{(ex)}$.

Solution: In this function also, since the variable is raised to a variable power therefore the differentiation can not be done using the ordinary rules of differentiation. To obtain a derivative of $y = x^{(ex)}$, on both sides of this equation we apply the natural logarithm as follows.

$$\ln y = \ln x^{(e^x)}$$

$$= e^x \ln x$$

Now differentiate both the sides of the obtained equation. For differentiation on the left-hand side is done using the chain rule because here y represents a function of x while the differentiation on the right-hand side is done using the product rule. Thus, to differentiate we start with the equation,

$$\ln y = e^x \ln x$$

On differentiating, we obtain the following equation:

$$\frac{1}{y}y' = e^x \left\{ \frac{1}{x} \right\} + e^x \ln x$$

Now first of all obtain a common denominator and then combine the fractions on the right-hand side as follows:

$$= \frac{e^x}{x} + e^x \ln x \left\{ \frac{x}{x} \right\}$$

$$= \frac{e^x}{x} + \frac{xe^x \ln x}{x}$$

$$= \frac{e^x + xe^x \ln x}{x}$$

Further, in the numerator factor out e^x as shown below.

$$= \frac{e^x(1 + x \ln x)}{x}$$

On multiplying both the sides of the above equation by y, we obtain:

$$y' = y \frac{e^x(1 + x \ln x)}{x}$$

$$= x^{(e^x)} \frac{e^x(1 + x \ln x)}{x^1}$$

On combining the powers of x we get the following resultant equation:

$$= x^{(e^x - 1)} e^x(1 + x \ln x)$$

Example 12. Differentiate $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$

Solution: In this function also, since the variable is raised to a variable power therefore the differentiation can not be done using the ordinary rules of differentiation.

To obtain a derivative of $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$, on both sides of this equation we apply the natural logarithm as follows.

$$\ln y = \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right)$$

$$= \ln\left(\sqrt{x}^{\sqrt{x}}\right) + \ln\left(e^{x^2}\right)$$

$$= \sqrt{x}\ln(\sqrt{x}) + x^2 \ln(e)$$

$$= \sqrt{x}\ln(\sqrt{x}) + x^2(1)$$

$$= \sqrt{x}\ln(\sqrt{x}) + x^2$$

Now differentiate both the sides of the obtained equation. For differentiation on the left-hand side is done using the chain rule because here *y* represents a function of *x* while the differentiation on the right-hand side is done using the product rule. Thus, to differentiate we start with the equation,

$$\ln y = \sqrt{x}\ln(\sqrt{x}) + x^2$$

On differentiating, we obtain the following equation:

$$\frac{1}{y}y' = \sqrt{x}\left\{\frac{1}{\sqrt{x}}\right\}(1/2)x^{-1/2} + (1/2)x^{-1/2}\ln(\sqrt{x}) + 2x$$

$$= \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x$$

Now first of all obtain a common denominator and then combine the fractions on the right-hand side as follows:

$$= \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x\left\{\frac{2\sqrt{x}}{2\sqrt{x}}\right\}$$

$$= \frac{1 + \ln(\sqrt{x}) + 4x^{1+1/2}}{2\sqrt{x}}$$

$$= \frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}}$$

On multiplying both the sides of the above equation by *y*, we obtain:

$$y' = y\frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}}$$

$$= \sqrt{x}^{\sqrt{x}} e^{x^2} \frac{1 + \ln(\sqrt{x}) + 4x^{3/2}}{2\sqrt{x}^1}$$

On combining the powers of $\sqrt{x}$ we get the following resultant equation:

$$= (1/2)\sqrt{x}^{(\sqrt{x}-1)} e^{x^2} \left\{1 + \ln(\sqrt{x}) + 4x^{3/2}\right\}$$

1.5 DIFFERENTIATION OF IMPLICIT FUNCTIONS

In an **implicit function** or relation the dependent variable is not inaccessible on one side of the equation, such as the equation $x^2 + xy - y^2 = 1$ denotes or signifies an implicit relation.

Mathematically, an implicit equation is defined as a relation of the form,

$R(x_1, \ldots, x_n) = 0$

Here *R* is stated as a function of several variables, essentially a polynomial.

The unit circle has the implicit equation of the form,

$$x^2 + y^2 - 1 = 0$$

Definition. Therefore, an **implicit function** is a **function** that is defined implicitly by an **implicit equation**, by associating one of the **variables (the value)** with the others (**the arguments**).

Consequently, for the unit circle an implicit function for *y* is defined implicitly by,

$$x^2 + f(x)^2 - 1 = 0$$

The above implicit equation states that *f* is a function of *x* only when,

$$-1 \le x \le 1$$

The values of the function are only non-negative or non-positive.

In the equations when *y* is not expressed explicitly in terms of *x* then implicit function is used. For example,

$$y^4 + 2x^2y^2 + 6x^2 = 7$$

In the above equation it is difficult to solve for *y* for finding *dy/dx*. To obtain the derivatives of such form of expressions for finding the rate of change of *y* as *x* changes we apply implicit differentiation method.

Example 13. Find the expression for $\dfrac{dy}{dx}$ if $y^4 + x^5 - 7x^2 - 5x^{-1} = 0$.

Solution: First we obtain the following expression:

$$y^4 + x^5 - 7x^2 - 5x^{-1} = 0$$

For obtaining the above expression, follow the steps given below.

(i) Finding the derivative with regard to *x* of y^4,

$$\frac{d}{dx}y = \frac{dy}{dx}$$

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$$

So therefore,

$$\frac{d}{dx}y^4 = 4y^3\frac{dy}{dx}$$

(ii) Finding the derivative with regard to *x* for,

$$x^5 - 7x^2 - 5x^{-1}$$

This simply refers to the ordinary differentiation

$$\frac{d}{dx}\left(x^5 - 7x^2 - 5x^{-1}\right) = 5x^4 - 14x + 5x^{-2}$$

(iii) Now we solve as follows.

To the right-hand side of the expression, we drive the following form that the derivative of zero is zero, i.e.,

$$\frac{d}{dx}(0) = 0$$

Let us combine the results of sections (i), (ii) and (iii) as follows:

$$4y^3 \frac{dy}{dx} + 5x^4 - 14x + 5x^{-2} = 0$$

Solving for *dy/dx* we obtain the following required expression:

$$\frac{dy}{dx} = \frac{-5x^4 + 14x - 5x^{-2}}{4y^3}$$

Check Your Progress

5. When do we use logarithmic differentiation?

6. What is the value of ln 1?

7. Define implicit function.

1.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Differentiation is the process of finding the derivative, or rate of change, of a function.

2. *f'(x)*.

3. A function that contains a parameter is termed as parametric function and the method of differentiating a parametric function is defined as the parametric differentiation.

4. Here, $dx/dt = 3$. Also, $dy/dt = 4$. Hence, $dy/dx = dy/dt/dx/dt = 4/3$.

5. This method is used in when it is easy to differentiate the logarithm of a function instead of the function itself.

6. In 1 = 0

7. An implicit function is a function that is defined implicitly by an implicit equation, by associating one of the variables with the others.

1.7 SUMMARY

- Differentiation is the process of finding the derivative, or rate of change, of a function.

- Derivatives are often used for finding the maxima and minima of a function and the equations which involve the derivatives are termed as differential equations.

- A tangent is defined as specific line which touches the curve only at one point.

- To find the derivative of an expression where one variable is the dependent variable generally named 'y' is typically expressed as a function of another independent variable generally named 'x', mathematically written as $y = f(x)$.

· $\Delta y = m \, \Delta x$, where 'm' defines the slope.

- The derivative of f at the point $x = a$ is the slope of the tangent line of the function f at the specified point a. This is frequently denoted as $f'(a)$ in Lagrange's notation and as $\dfrac{dy}{dx}|_{x=a}$ in Leibniz's notation.

- In the domain of 'f' there is derivative for each point 'a' which defines a function for sending every specified point 'a' to the derivative of 'f' at 'a', such as when $f(x) = x^2$, then at that point the derivative function is
$$f'(x) = \frac{dy}{dx} = 2x.$$

- A function that contains a parameter is termed as parametric function and the method of differentiating a parametric function is defined as the parametric differentiation.

- A parametric derivative is approximated using a derivative of a dependent variable 'y' with reference to an independent variable 'x' specifically taken in the condition when both the variables typically depend on an third independent variable 't', normally assumed as 'time', i.e., specifically when the variables x and y are defined through the parametric equations in t.

- Parametric differentiation: if $x = x(t)$ and $y = y(t)$ then $dy/dx = (dy/dt)/(dx/dt)$ where dx/dt is not equal to zero.

- Logarithmic differentiation a specific method to differentiate functions by means of the logarithmic derivative of a function 'f'.

- The logarithmic principle is applied in the differentiation of all differentiable functions provided that the functions are non-zero.

- An implicit function is a function that is defined implicitly by an implicit equation, by associating one of the variables (the value) with the others (the arguments).

1.8 KEY WORDS

- **Derivative:** Derivative, in mathematics, the rate of change of a function with respect to a variable.

- **Maxima and minima:** In mathematical analysis, the maxima and minima of a function, known collectively as extrema, are the largest and smallest

value of the function, either within a given range or on the entire domain of a function.

- **Slope:** The slope of a line is the ratio of the amount that y increases as x increases some amount. Slope tells you how steep a line is, or how much y increases as x increases. The slope is constant (the same) anywhere on the line.

- **Logarithm:** a quantity representing the power to which a fixed number (the base) must be raised to produce a given number.

1.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write two applications of differentiation.
2. Define the terms rate of change and slope.
3. Why do we need a parameter in differentiation?
4. Find the derivative of the parametric function when $x = 9t^2$ and $y = 5t^3$.

Long Answer Questions

1. Given are the following parametric equations:

   $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$. Plot a graph for this.

2. Explain differentiation by using logarithms.
3. Discuss properties of natural logarithms.
4. Differentiate $y = 5x^{(e^{2x})}$.
5. Find the expression for $\dfrac{dy}{dx}$ if $y^3 + x^3 - 2x^2 - 27 = 0$

1.10 FURTHER READINGS

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UNIT 2 SUCCESSIVE DIFFERENTIATION AND STANDARD FUNCTIONS

Structure

2.0 INTRODUCTION

In the previous unit, you have learned the differentiation and different methods to differentiate a function. This unit introduces you to the concept of successive differentiation. In successive differentiation, a function can be differentiated more than once. Further in this unit, you will see *n*th derivative of some standard functions. In the end, some problems are discussed using higher order derivatives. The derivatives other than the first derivative are called the higher order derivatives.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of successive differentiation
- Know the *n*th derivative of some standard functions
- Solve problems using higher order derivatives

2.2 SUCCESSIVE DIFFERENTIATION - INTRODUCTION

The **successive differentiation** refers to the method of finding the **derivatives** in **succession** or more precisely in sequence.

In successive differentiation, a function can be differentiated more than once as explained here.

As a general rule, the differential coefficient of a function $f(x)$ is itself a function of x specified as derived function of x or the first derivative of the function $f(x)$. The first derivative of the function $f(x)$ is symbolized or represented as $f'(x)$ which indicates that function $f(x)$ is differentiated first time and is defined as the first derivative. This first derivative $f'(x)$ is then differentiated for obtaining the second derivative of the function $f(x)$ and is symbolized or represented as $f''(x)$ specifying that the function $f(x)$ is differentiated twice. When the derived function $f'(x)$ is differentiated second time then we can differentiate the obtained second derivative $f''(x)$ third time to provide a function of x termed as third derivative.

This second derivative $f''(x)$ is then differentiated for obtaining the third derivative of the function $f(x)$ and is symbolized or represented as $f'''(x)$ specifying that the function $f(x)$ is differentiated three times. This method of obtaining a derived function can be repeated indefinitely and the derived function thus obtained is symbolized or represented as $f^{(n-2)}(x)$.

The derived function $f^{(n-2)}(x)$ is differentiated then we obtain the $(n-1)$th derivative $f^{(n-1)}x$ which can be further differentiated to provide a function of x termed as nth derivative and is represented or symbolized as $f^n(x)$ specifying that $f(x)$ is differentiated n-times.

This method of differentiation of a function $f(x)$ is successively repeated more than once to find derivatives sequentially from a given function $f(x)$ is termed as **successive differentiation**.

Therefore, if y is considered as a function of x which can be differentiated with regard to x, then the obtained derivative dy/dx is termed as the derivative of first order. Moreover if this first derivative dy/dx is used as a differentiable function then we can further differentiate it with reference to x. This second derivative of y with reference to x is represented or symbolized as d^2y/dx^2.

Additionally, if the second derivative d^2y/dx^2 is further differentiated then we obtain the third derivative of y which is represented or symbolized as d^3y/dx^3. In the same way, the nth derivative of y is represented or symbolized as d^ny/dx^n.

Mathematically, all the obtained derivatives of y with respect to x are termed as **successive derivatives** and the method is specified as **successive differentiation**.

The successive derivatives are used in approximating,

1. Maxima and Minima of Functions

2. Increasing and Decreasing Functions

3. Determining the Points of Inflexion while Curve Tracing

4. Concavity and Convexity of Curves at Specific Points

Hence, for $y = f(x)$ if there exist its successive derivatives upto the ***n*th** order then we have the following forms of derivatives,

$$\frac{dy}{dx}, \; \frac{d^2y}{dx^2}, \; \frac{d^3y}{dx^3} \cdots \cdots, \; \frac{d^ny}{dx^n}$$

These derivatives are termed as **successive derivatives** or more specifically **differential coefficients** of *y* with reference to *x*. Additionally, the following forms are also used to represent or symbolize these successive derivatives or differential coefficients:

$$y', \; y'', \; y''', \ldots., \; y^{(n)}, \ldots.$$
$$y_1, \; y_2, \; y_3, \ldots., \; y_n, \ldots$$
$$f'(x), \; f''(x), \; f'''(x), \ldots., \; f^{(n)}(x), \ldots$$
$$Df(x), \; D^2f(x), \; D^3f(x), \ldots., \; D^nf(x), \ldots.$$

Therefore, for $y = f(x)$ the derivative $f^{(n)}(x)$ can also be represented as d^ny/dx^n or D^ny or $y^{(n)}$ or y_n, and also as,

$$f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

Derivation of Derivative Equation

If $y = f(x)$ is considered as the **differentiable function** of *x*, then we have the following form termed as the **first differential coefficient** of *y* with respect to *x*:

$$\frac{dy}{dx} = f'(x)$$

Now, on differentiating both side with respect to *x*,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{d}{dx}\right)[f'(x)] = f''(x).$$

Consider that if, $\left(\dfrac{d}{dx}\right)\left(\dfrac{dy}{dx}\right)$ is denoted by,

$$\frac{d^2y}{dx^2};$$

Then,

$$\frac{d^2y}{dx^2} = f''(x)$$

In the same way, $\left(\dfrac{d}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)$ is denoted by,

$$\frac{d^3 y}{dx^3};$$

To be precise, $\frac{d^3 y}{dx^3} = f'''(x)$, and so on.

Consequently, the following expressions are termed as **first**, **second**, **third**, ..., **nth differential coefficient** of '*y*'.

$$\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{d^3 x}, \ldots \ldots \frac{d^n y}{dx^n}$$

Generally these functions can also be written as follows:

$$y', y'', y''', \ldots \ldots y^n$$

Or,

$$y_1, y_2 y_3 \ldots \ldots \ldots \ldots \ldots y_n$$

And also,

$$Dy, D^2 y, D^3 y, \ldots \ldots D^n y.$$

(Where $D = d/dx$, or capital D notation and n being the positive integer.)

$$\frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \quad \frac{d^3 y}{dx^3}, \quad \ldots \ldots \quad \frac{d^n y}{dx^n}, \ldots \ldots \ldots$$

$$f'(x), \quad f''(x), \quad f'''(x), \ldots \ldots \ldots, f^n(x), \ldots \ldots$$

Additionally, for $y = f(x)$, the *n*th order derivative at $x = a$ is normally denoted or represented by the form,

$$\left(\frac{d^n y}{dx^n} \right)_{x=a}$$

Or, $(y_n)_{x=a}$ or $(y^n)_{x=a}$ or $f^n(a)$

Example 1. Successively differentiate $y = x^6$ up to fourth derivative.

Solution: Given is, $y = x^6$.

Now we will successively differentiate $y = x^6$ as follows.
The first derivative will be,

$$y_1 = \frac{dy}{dx} = 6x^{(6-1)} = 6x^5$$

The second derivative will be,

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2}$$

$$= 6 \times 5 \times x^{(5-1)} = 30x^4$$

The third derivative will be,

$$y_3 = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^3y}{dx^3}$$

$$= 30 \times 4 \times (x)^{(4-1)} = 120 \cdot x^3$$

Similarly, the fourth derivative will be,

$$y_4 = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

$$= \frac{d^4y}{dx^4}$$

$$= 120 \times 3 \times (x)^{(3-1)} = 360 \cdot x^2$$

Example 2. Successively differentiate $y = \sin x$ up to nth derivative.

Solution: Given is, $y = \sin x$

Therefore,

$$y_1 = \cos x$$

$$= \sin\left(\frac{\pi}{2} + x\right)$$

$$y_2 = -\sin x$$

$$= \cos\left(\frac{\pi}{2} + x\right) = \sin\left(2 \cdot \frac{\pi}{2} + x\right)$$

$$y_3 = -\cos x$$

$$= \cos\left(2 \cdot \frac{\pi}{2} + x\right) = \sin\left(3 \cdot \frac{\pi}{2} + x\right)$$

Similarly, we can obtain the values for $y_4 = \sin x$, $y_5 = \cos x$, and so on. And the notation for nth derivative will be,

$$y_n = \sin\left(n \cdot \frac{\pi}{2} + x\right)$$

Check Your Progress

1. What do you understand by successive differentiation?

2. How the third derivative of a function $f(x)$ is donated?

3. What is the first derivative of $\sin x$?

2.3 nTH DERIVATIVE OR DIFFERENTIAL COEFFICIENT OF SOME STANDARD FUNCTIONS

Following are some standard functions of the nth derivative or differential coefficient.

(i) Differential Coefficient of x^m

For, $y = x^m$

We have,

$y_1 = mx^{m-1}$

$y_2 = m(m-1) x^{m-2}$

$y_3 = m(m-1)(m-2) x^{m-3}, \ldots$, and so on.

And, $Y_n = m(m-1)(m-2)(m-3)(m-4) \ldots \ldots (m-n+1) x^{m-n}$

Further, if m is a positive integer then,

$Y_n = 1.2.3. \ldots \ldots m = m!$

Therefore,

$D_n(x^m) = m(m-1)(m-2)(m-3)(m-4) \ldots \ldots (m-n+1) x^{m-n}$

(ii) Differential Coefficient of $(ax + b)^m$

For, $y = (ax + b)^m$

We have,

$y_1 = am(ax + b)^{m-1}$

$y_2 = a^2 m(m-1)(ax + b) x^{m-2}$

$y_3 = a^3 m(m-1)(m-2)(ax + b) x^{m-3}, \ldots$, and so on.

And,

$y_n = a^n m(m-1)(m-2)(m-3)(m-4) \ldots \ldots (m-n+1) (ax + b) x^{m-n}$

Therefore,

$D(ax + b)^m = a^n m(m-1)(m-2)(m-3) \ldots \ldots (m-n+1)(ax + b)^{m-n}$

Or,

$$D(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

For *m* being the negative integer, take $m = -p$, here *p* is defined as the positive integer. Accordingly we can have,

$$D(ax+b)^{-p} = a^n(-p)(-p-1)(-p-2)\ldots\ldots\ldots\ldots[-p-(n-1)](ax+b)^{-p-n}$$

$$= a^n(-1)^n p(p+1)(p+2)\ldots\ldots\ldots\ldots[(p-n+1)](ax+b)^{-p-n}$$

$$= (-1)^n \frac{(p-n+1)!}{(p-1)!}a^n(ax+b)^{-p-n}$$

Remember that,

(a) For **m = n**, we have,

$$D^n (ax+b)^{-p} = a^n !$$

(b) For **m = – 1**, we have,

$$D^n(ax+b)^{-1} = (-1)(-1-1)\ldots\ldots\ldots(-1-n+1)a^n(ax+b)^{-1-n}$$

$$(-1)^n 1.2.3\ldots\ldots\ldots na^n(ax+b)^{-1-n}$$

$$= (-1)^n n! a^n (ax+b)^{-1-n} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}.$$

(iii) Differential Coefficient of (*ax* + *b*)

For $y = \log(ax+b)$

We have,

$$y_1 = \frac{a}{(ax+b)} = a(ax+b)^{-1} = \frac{a(0!)}{(ax+b)}$$

$$y_2 = \frac{a^2.1}{(ax+b)^2} = -\frac{a^2.(1!)}{(ax+b)^2}$$

$$y_3 = \frac{a^3.2}{(ax+b)^3} = -\frac{a^3.(2!)}{(ax+b)^3}$$

$$y_4 = \frac{a^4.2.3}{(ax+b)^4} = (-1)^3.\frac{a^4.(3!)}{(ax+b)^4}, \text{ and so on.}$$

And typically,

$$y_n = (-1)^{n-1}.\frac{a^n.(n-1)!}{(ax+b)^n}$$

Therefore,

$$D^n \log(ax+b) = (-1)^{n-1}.\frac{a^n.(n-1)!}{(ax+b)^n}.$$

Remember that,

$$D^n \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

(iv) Differential Coefficient of a^{bx}

For $y = a^{bx}$

We have,

$$y_1 = ba^{bx} \log a$$

$$y_2 = b^2 a^{bx} (\log a)^2.$$

$$y_3 = b^3 a^{bx} (\log a)^3, \text{ and so on.}$$

And also,

$$y_n = b^n a^{bx} (\log a)^n$$

Therefore,

$$D^n . a^{bx} = b^n a^{bx} (\log_e a)^n.$$

(v) Differential Coefficient of e^{ax}

For $y = e^{ax}$

We have,

$y_1 = ae^{ax}$

$y_2 = a^2 e^{ax}$

$y_3 = a^3 e^{ax}$

$y_4 = a^4 e^{ax}$

$y_5 = a^5 e^{ax}$, and so on.

And also,

$y_n = a^n e^{ax}$

Therefore,

$$D^n e^{ax} = a^n e^{ax}$$

(vi) Differential Coefficient of sin $(ax + b)$

For $y = \sin (ax + b)$

We have,

$$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_2 = -a^2\cos(ax+b) = a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right], \ldots, \text{ and so on.}$$

And also,

$$y_n = a^n\sin\left(ax+b+\frac{1}{2}n\pi\right)$$

Therefore,

$$D^n(ax+b) = a^n\sin\left[ax+b+\frac{1}{2}n\pi\right]$$

Remember that,

$$D^n\sin x = \sin\left[x + \left(\frac{n\pi}{2}\right)\right]$$

(vii) Differential Coefficient of cos (*ax* + *b*)

For $y = \cos(ax+b)$
We have,

$$y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2} + ax+b\right)$$

$$y_2 = -a^2\sin\left(\frac{\pi}{2} + ax+b\right) = a^2\cos\left(\frac{2\pi}{2} + ax+b\right), \ldots, \text{ and so on.}$$

And also,

$$y_n = a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$$

Therefore,

$$D^n\cos x(ax+b) = a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$$

Remember that,

$$D^n\cos x = \cos\left(x + \frac{1}{2}n\pi\right).$$

(viii) Differential Coefficient of $e^{ax} \sin(bx + c)$ and $e^{ax} \cos(bx + c)$

For $y = e^{ax} \sin(bx + c)$

We have,

$$y_1 = e^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$$

$$= e^{ax}[a\sin(bx + c) + b\cos(bx + c)]$$

Now when we put, $\quad a = r\cos\phi \quad$ and $\quad b = r\sin\phi$

Then we have,

$y_1 = re^{ax} \sin(bx + c + \phi)$

In which, $\quad r^2 = a^2 + b^2 \quad$ and $\quad \phi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

In the same way,

$y_2 = r^2 e^{ax} \sin(bx + c + 2\phi), \ldots \ldots \ldots$, and so on.

Therefore,

$$D^n \, e^{ax} \sin(bx + c) = r^n e^{ax} \sin(bx + c + n\phi)$$

In which,

$$r = (a^2 + b^2)^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\dfrac{b}{a}\right)$$

In the same way,

$D^n \, e^{ax} \cos(bx + c) = r^n e^{ax} \cos(bx + c + n\phi)$

In which,

$$r = (a^2 + b^2)^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\dfrac{b}{a}\right).$$

Example 3. Evaluate the nth derivative of $e^{ax} \sin bx \cos cx$.

Solution: The nth derivative of $e^{ax} \sin bx \cos cx$ is evaluated as follows.

Consider that,

$$y = e^{ax} \sin bx \cos cx = \frac{1}{2} e^{ax}(2\sin bx \cos cx)$$

$$= \frac{1}{2} e^{ax}[\sin(bx + cx) + \sin(bx - cx)]$$

$$= \frac{1}{2}[e^{ax} \sin(b + c)x + e^{ax} \sin(b - c)x].$$

And also,

$$D^n\left[e^{ax}\sin(bx+cx)\right]=(b^2+c^2)^{n/2}e^{ax}\sin\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Therefore,

$$y^n=\frac{1}{2}\left[\begin{array}{l}\left\{a^2+(b+c)^2\right\}^{n/2}e^{ax}\sin\left\{(b+c)x+n\tan^{-1}(b+c)/a\right\}\\+\left\{a^2(b-c)^2\right\}^{n/2}e^{ax}\sin\left\{(b-c)x+n\tan^{-1}\dfrac{(b-c)}{a}\right\}\end{array}\right]$$

Example 4. For $y=e^{ax}$, evaluate $\dfrac{d^ny}{dx^n}$

Solution: The $\dfrac{d^ny}{dx^n}$ is evaluated as follows.

We know that,

$$\frac{dy}{dx}=ae^{ax}$$

Further,

$$\frac{d^2y}{dx^2}=a^2e^{ax}$$

Therefore,

$$\frac{d^ny}{dx^n}=a^ne^{ax}.$$

Example 5. If $y=\sin x$, then find $\dfrac{d^ny}{dx^n}$

Solution: We can find $\dfrac{d^ny}{dx^n}$ as follows.

Since,

$$\frac{dy}{dx}=\cos x=\sin\left(x+\frac{\pi}{2}\right)$$

And,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right)=\cos\left(x+\frac{\pi}{2}\right)=\sin\left(x+\frac{2\pi}{2}\right).$$

$$\frac{d^3 y}{dx^3} = \frac{d}{dx} \sin\left(x + \frac{2\pi}{2}\right) = \cos\left(x + \frac{2\pi}{2}\right) = \sin\left(x + \frac{3\pi}{2}\right),$$

. . . ., and so on.
Therefore,

$$\frac{d^n y}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right).$$

2.4 PROBLEMS USING HIGHER ORDER DERIVATIVES

Together the **second**, **third**, **fourth**, ..., etc., **derivatives** are termed as the **higher order derivatives**.

Since the derivative of a function $y = f(x)$ is a function itself and its derivative is represented as $y' = f'(x)$. The derivative of $f'(x)$ is normally termed as the *second derivative of f(x)* and is denoted as $f''(x)$ or $f^2(x)$. The successive differentiation method is continued for further finding the third, fourth, fifth, . . . , and so on, the successive derivatives of $f(x)$, termed as the **higher order derivatives** of $f(x)$. The **'prime'** notation used for derivatives may sometimes create confusion hence the numerical notation $f^{(n)}(x) = y^{(n)}$ is preferably used for denoting the nth derivative of $f(x)$.

Take the following function into consideration:

$$f(x) = 5x^3 - 3x^2 + 10x - 5$$

On differentiating this function, we have,

$$f'(x) = 15x^2 - 6x + 10$$

On further differentiating this function, we have the derivative,

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

This derivative is termed as the **second derivative** while $f'(x)$ is termed as the **first derivative**.

On further differentiating this function we obtain the **third derivative**,

$$f'''(x) = \left(f''(x)\right)' = 30$$

Similarly, on differentiating this function we obtain the **fourth derivative**,

$$f^{(4)}(x) = \left(f'''(x)\right)' = 0$$

In the above notation, prime is not used, instead numerical notation is used because adding too many primes will make the notation cumbersome.

Example 5. Evaluate the first, second and third derivatives of $y = \sin^2 x$.

Solution: Given is $y = \sin^2 x$

Therefore, we obtain the first, second and third derivatives as follows.

$y' = 2 \sin x \cos x$

$y'' = 2 \cos x \cos x + 2 \sin x (-\sin x)$

$\quad = 2 \cos^2 x - 2 \sin^2 x$

$y''' = 2 \cdot 2 \cos x (-\sin x) - 2 \cdot 2 \sin x \cos x$

$\quad = -4 \sin x \cos x - 4 \sin x \cos x$

$\quad = -8 \sin x \cos x$

Example 6. If $x^2 + y^4 = 10$, then find y''.

Solution: Given is $x^2 + y^4 = 10$. We can find y'' or second derivative as follows.

We have to find the first derivative,

$$2x + 4y^3 y' = 0$$
$$y' = -\frac{x}{2y^3}$$

Therefore, the first derivative is,

$$y' = -\frac{x}{2y^3}$$

Now we obtain the second derivative y'' as follows:

$$y'' = \left(-\frac{x}{2y^3}\right)'$$
$$= -\frac{2y^3 - x\left(6y^2 y'\right)}{\left(2y^3\right)^2}$$
$$= -\frac{2y^3 - 6xy^2 y'}{4y^6}$$
$$= -\frac{y - 3xy'}{2y^4}$$

Substituting the first derivative into the second derivative equation, we find y'' as follows,

$$y'' = -\frac{y - 3xy'}{2y^4}$$
$$= -\frac{y - 3x\left(-\frac{x}{2y^3}\right)}{2y^4}$$
$$= -\frac{y + \frac{3}{2}x^2 y^{-3}}{2y^4}$$

Example 7. For $f(x) = (3 - 5x)^5$ find $f'''(x)$.

Solution: Given is,

$$f(x) = (3 - 5x)^5$$

Now we will find $f'''(x)$ by obtaining the first, second and third order derivatives as follows.

$$f'(x) = 5 (3 - 5x)^4 (- 5) = - 25 (3 - 5x)^4$$
$$f''(x) = - 25 (4) (3 - 5x)^3 (- 5) = 500 (3 - 5x)^3$$
$$f'''(x) = 500 (3) (3 - 5x)^2 (- 5) = - 7500 (3 - 5x)^2$$

Example 8. Find the first four derivatives for $y = \cos x$.

Solution: Given is that $y = \cos x$. We will find the first four derivatives as follows.

First derivative $\qquad\qquad y' = - \sin x$

Second derivative $\qquad\qquad y'' = - \cos x$

Third derivative $\qquad\qquad y''' = \sin x$

Fourth derivative $\qquad\qquad y'''' = \cos x$

Check Your Progress

4. What is the differential coefficient of x^m?

5. What are higher order derivatives?

2.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The successive differentiation refers to the method of finding the derivatives in succession. In successive differentiation, a function can be differentiated more than once.

2. $f'''(x)$.

3. $\cos x$.

4. $D_n (x^m) = m(m - 1) (m - 2) (m - 3) (m - 4) \ldots\ldots\ldots (m - n + 1) x^{m-n}$

5. Together the second, third, fourth, ..., etc., derivatives are termed as the higher order derivatives.

2.6 SUMMARY

- The successive differentiation refers to the method of finding the derivatives in succession. In successive differentiation, a function can be differentiated more than once.

- If y is considered as a function of x which can be differentiated with regard to x, then dy/dx is first derivative, d^2y/dx^2 is second derivative, d^3y/dx^3 is third derivative and $d^n y/dx^n$ is nth derivative.

- All the obtained derivatives of y with respect to x are termed as successive derivatives and the method is specified as successive differentiation.

- Differential Coefficient of x^m, $D_n(x^m) = m(m-1)(m-2)(m-3)(m-4) \ldots\ldots\ldots (m-n+1)\, x^{m-n}$

- Differential Coefficient of $(ax+b)^m$,

$$D(ax+b)^m = a^n m(m-1)(m-2)(m-3)\ldots\ldots\ldots(m-n+1)(ax+b)^{m-n}$$

- Differential Coefficient of $(ax+b)$,

$$D^n \log(ax+b) = (-1)^{n-1} \cdot \frac{a^n \cdot (n-1)!}{(ax+b)^n}.$$

- Differential Coefficient of a^{bx}

$$D^n \cdot a^{bx} = b^n a^{bx} (\log_e a)^n.$$

- Differential Coefficient of e^{ax}

$$D^n\, e^{ax} = a^n e^{ax}$$

- Differential Coefficient of $\sin(ax+b)$,

$$D^n(ax+b) = a^n \sin\left[ax + b + \frac{1}{2}n\pi \right]$$

- Differential Coefficient of $\cos(ax+b)$

$$D^n \cos x(ax+b) = a^n \cos\left(ax + b + \frac{1}{2}n\pi \right)$$

2.7 KEY WORDS

- **Derivative:** Derivative, in mathematics, the rate of change of a function with respect to a variable.

- **Coefficient:** a numerical or constant quantity placed before and multiplying the variable in an algebraic expression (e.g. 4 in $4xy$).

- **Differentiation:** Differentiation, in mathematics, process of finding the derivative, or rate of change, of a function.

2.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on successive differentiation.
2. What is the importance of successive derivatives?
3. Explain nth order derivative for $y = f(x)$.
4. Find the first four derivatives for $y = \sin x$.

Long Answer Questions

1. Evaluate the first, second and third derivatives of $y = \cos^2 x$.

2. If $y = \cos x$ then find $\dfrac{d^n y}{dx^n}$.

3. If $x^2 - y^4 = 10$, then find y'''.

4. For $f(x) = (5x + x)^5$ find $f'''(x)$.

2.9 FURTHER READINGS

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Narayanan, S, T.K. Manickavasagam Pillai. 2009. *Differential Equations and its applications*. Chennai: S.Viswanathan(Printers & Publishers) Pvt. Ltd.

Venkataraman, M.K, Manorama Sridhar. *Calculus and Fourier series*. Chennai: The National Publishing Company.

UNIT 3   PARTIAL DIFFERENTIATION

Structure

3.0   INTRODUCTION

You are already aware of the concept of differentiation and successive differentiation. In this unit, you will learn the concept of partial differentiation and its applications. In mathematics, a partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary). Partial derivatives are used in vector calculus and differential geometry. Partial derivatives play a prominent role in economics, in which most functions describing economic behaviour proposes that the behaviour depends on more than one variable.

You will also learn about the homogeneous functions and partial derivatives of homogeneous functions. A homogeneous function is one with multiplicative scaling behaviour: if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor. Further this unit discusses Euler's theorem, which is extremely useful in differential calculus. In the end, this unit explains the properties of maxima and minima of functions of one variable and two variables.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of partial differentiation
- Know about homogeneous function and results based on it
- Discuss Euler's theorem and its verification
- Determine the maxima and minima of functions of one variable and two variable

3.2 PARTIAL DIFFERENTIATION

Assume that for a function of two variables, $f(x, y)$, basically the derivative of f is defined simply with respect to x while y is considered as a constant. This derivative is termed as the "**partial derivative of f with respect to x**" and is represented either in terms of $\partial f / \partial x$ or f_x. The '∂' symbol is used for denoting partial derivative.

Likewise, we can also define the derivative of f simply with respect to y while x is considered as a constant. This derivative is termed as the "**partial derivative of f with respect to y**" and is represented either in terms of $\partial f / \partial y$ or f_y.

Definition. The **partial derivative** of a **function** $f(x, y, \dots)$ with reference to the variable x is represented using the following specified notations:

$$f'x, \quad fx, \quad \partial_x f, \quad D_x f, \quad D_1 f, \quad \partial f / \partial x, \quad \text{or} \quad \frac{\partial}{\partial x} f$$

In addition, for a function $z = f(x, y, \dots)$, the partial derivative of z with regard to x is represented by $\partial z / \partial x$.

Generally, the arguments of a partial derivative is same as the original function, however, occasionally its functional dependency is explicitly denoted or represented by the notation as shown below.

$$f_x(x, y, \dots), \quad \frac{\partial f}{\partial x}(x, y, \dots).$$

Definition. Mathematically, a **partial derivative** of a **function** of **several variables** is its derivative with respect to one of those variables, while the remaining others are considered as constant. Partial derivatives has its applications in vector calculus, differential geometry, etc.

Definition. The function f can be interpreted or deduced as a family of functions of one variable which is indexed by the other variables as,

$$f(x, y) = f_y(x) = x^2 + xy + y^2$$

Definition. A differential equation which expresses one or more measures in terms of partial derivatives is called a partial differential equation.

(1) Partial Derivative, the **Function** of **One Variable (x)**:

$$f(x) = x^2$$

Its **derivative** by means of the **Power Rule** is:

$$f'(x) = 2x$$

In calculus, the Power Rule is frequently used to find the derivative.

As per the Power Rule, the **derivative** of x^n is $nx^{(n-1)}$

Example 1. Find the derivative of x^2 using the Power Rule.

Solution: To find the derivative of x^2 by means of Power Rule using $n = 2$, follow the given method.

The derivative for,

$$x^2 = 2x^{(2-1)}$$
$$= 2x^1$$
$$= 2x$$

Thus, the derivative of x^2 is $2x$.

(2) Partial Derivative, the **Function** of **Two Variables (x and y)**:

$$f(x, y) = x^2 + y^3$$

(i) For finding the partial derivative of the above expression of two variables with regard to x, we consider y as constant and evaluate the derivative as follows:

$$f'x = 2x + 0 = 2x$$

Interpretation: Let us understand the process.

(a) The derivative of x^2 with regard to x is defined as $2x$.

(b) Here, since y is considered as a constant, hence y^3 will also be a constant and consequently the derivative of a constant will be 0 (zero).

(ii) For finding the partial derivative of the above expression of two variables with regard to y, we consider x as constant and evaluate the derivative as follows:

$$f'y = 0 + 3y = 3y^2$$

Interpretation: Let us understand the process.

(a) The derivative of y^3 with regard to y is defined as $3y^2$.

(b) Here, since x is considered as a constant, hence x^2 will also be a constant and consequently the derivative of a constant will be 0 (zero).

How a Variable Constant is Hold as a Constant in Case of Two Variables

Let us understand the concept with the help of the following example of cylinder where we will consider the variables r (radius of cylinder) and h (height of cylinder) as a contant, each at a time (Refer Figure 3.1).
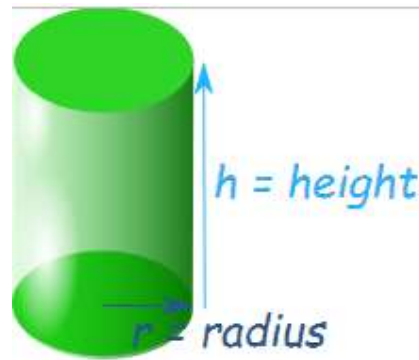
Fig. 3.1 *Radius and Height of a Cylinder*

We know that the volume of a cylinder is evaluated by,

$V = \pi\ r^2\ h$

Now in variable form,

$V = f(r, h)$

Hence,

$f(r, h) = \pi\ r^2\ h$

For defining the partial derivative with reference to *r* we consider *h* as a constant, as variable *r* changes (Refer Figure 3.2):

$f'r = \pi\ (2r)\ h = 2\ \pi\ r\ h$
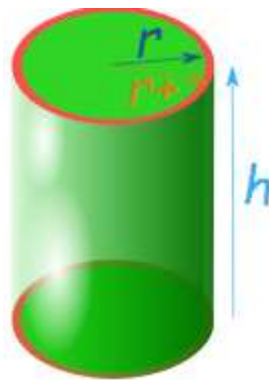
Fig. 3.2 *Radius 'r' Changes*

Therefore, the derivative of r^2 with reference to *r* is 2*r*, where π and *h* are considered as constants.

Thus it states that, "in this case the radius *r* only changes by the smallest amount and accordingly the volume of the cylinder changes by $2\ \pi\ r\ h$". The change in the radius is very less as shown in the Figure 3.2 and it seems as if it is a membrane having a circle circumference ($2\ \pi\ r$) and a height *h*.

For defining the partial derivative with reference to *h* we consider *r* as a constant, (Refer Figure 3.3):

$$f'h = \pi r^2 (1) = \pi r^2$$
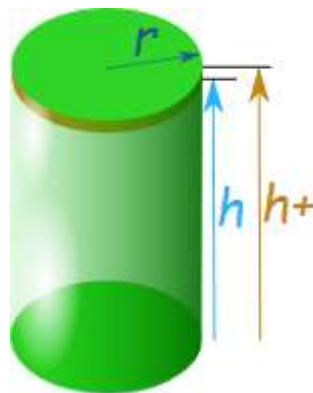
Fig. 3.3 Height 'h' Changes

Therefore, the derivative of *h* with reference to *h* is 1, and where π and r^2 are considered as constants.

Thus it states that, "In this case the height *h* only changes by the smallest amount and accordingly the volume of the cylinder changes by r^2". The change in the height is very less as shown in the Figure 3.3 and it seems as if a thin disk is added on the top having a circle area of πr^2.

Notations

(i) The partial derivative of '*f*' with respect to '*x*' is expressed as,

$$f_x \quad \text{or} \quad \partial f / \partial x$$

(ii) The partial derivative of '*f*' with respect to '*y*' is expressed as,

$$f_y \quad \text{or} \quad \partial f / \partial y$$

(iii) The analogous notation to the most familiarized form is expressed as,

$$df / dx \quad \text{or} \quad f'$$

(iv) For $z = f(x, y)$.

We can also write the partial derivative $f_x (x, y)$ as,

$$\frac{\partial f}{\partial x}(x, y) \text{ or } \frac{\partial z}{\partial x}$$

In the same way, we can write the partial derivative $f_y (x, y)$ as,

$$\frac{\partial f}{\partial y}(x, y) \text{ or } \frac{\partial z}{\partial y}.$$

(v) We can evaluate the partial derivative $f_y(x, y)$ at the point (x_0, y_0) and express as follows:

$$f_x(x_0, y_0), \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$$

Or, $\quad \dfrac{\partial f}{\partial x}(x_0, y_0)$

Try finding the equivalent expression for $f_y(x_0, y_0)$.

We have previously discussed that for a function of one variable $f(x)$, characteristically the derivative $f'(x)$ denotes the rate of change of the function when x changes. This is significant interpretation of derivatives. Let us understand the concept with the help of Example 2.

Example 2. Consider the function $f(x, y) = 2x^2y^3$ and then determine the rate of change by which the function is changing at the specified point (a, b),

(i) When y is constant and x is a varying variable.

(ii) When x is constant and y is a varying variable.

Solution: To determine the rate of change by which the function $f(x, y) = 2x^2y^3$ is changing at the specified point (a, b), let us first consider the case when y is constant and x is a varying.

Hence, we have $y = b$, because y is a constant.

Therefore, then there be a function of the form which will include only 'x' and can be defined as follows:

$$g(x) = f(x, b) = 2x^2b^3$$

At this time, this function has only one variable and for determining the rate of change we take, $g(x)$ at $x = a$.

Alternatively, we evaluate $g'(a)$, the function of one or single variable.

Accordingly the rate of change of the function at (a, b), when x is varying and y is considered as a constant, is evaluated as follows:

$g'(a) = 4ab^3$

The notation **$g'(a)$** is termed as the **partial derivative** of **$f(x, y)$** with reference to **x** at **(a, b)** and is represented as follows.

$f_x(a, b) = 4ab^3$

Again, let us now consider the case where the variable y is varying and x is a constant. Similarly as we can derive the equation as we have derived in the case when the variable x is varying and y is a constant.

Therefore, when x is a constant then it is constant at $x = a$.

Therefore, then there be a function of the form which will include only '**y**' and can be defined as follows:

$$h(y) \;=\; f(a, y) \;=\; 2a^2 y^3$$

$$\Rightarrow \qquad h'(b) \;=\; 6a^2 b^2$$

The notation **h'(b)** is termed as the **partial derivative** of **f(x, y)** with reference to **y** at **(a, b)** and is represented as follows.

$$f_y\,(a, b) = 6a^2 b^2$$

The **two partial derivatives** that we have derived in this **Example 2** are also occasionally termed as the **first order partial derivatives**.

Generally, for partial derivatives the notation (a, b) is not always used, instead for partial derivatives we use the standard notation (x, y). Hence, the partial derivatives of Example 2 can be written as follows using the standard notation (x, y):

$$f_x\,(x, y) = 4xy^3 \qquad \text{and} \qquad f_y\,(x, y) = 6x^2 y^2$$

Basically, to evaluate $f_x\,(x, y)$ we consider 'y' as constant for differentiating 'x' and similarly to evaluate $f_y\,(x, y)$ we consider 'x' as constant for differentiating 'y'.

Using the limit definition we can write the above two partial derivatives in limit form as follows:

$$f_x\,(x, y) = \lim_{h \to 0} \frac{f\,(x + h, y) - f\,(x, y)}{h}$$

$$f_y\,(x, y) = \lim_{h \to 0} \frac{f\,(x, y + h) - f\,(x, y)}{h}$$

Following are some precise and probable alternate notations to evaluate partial derivatives.

For the function $z = f(x, y)$ we have the following equivalent notations:

$$f_x\,(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\,(f\,(x, y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$

$$f_y\,(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\,(f\,(x, y)) = z_y = \frac{\partial z}{\partial y} = D_y f$$

The following fractional notations of partial differentiation specify that how the partial derivative is different from the ordinary derivative with reference to single variable calculus.

$$f\,(x) \qquad\qquad \Rightarrow \qquad\qquad f'\,(x) = \frac{df}{dx}$$

$$f\,(x, y) \qquad\qquad \Rightarrow \qquad\qquad f_x\,(x, y) = \frac{\partial f}{\partial x} \;\&\; f_y\,(x, y) = \frac{\partial f}{\partial y}$$

In the following examples, all the remaining variables are considered as a constant value and then we differentiate the derivative taking it as a function of single variable.

Example 3. For the following given functions, find all the first order partial derivatives.

(i)　　$f(x, y) = x^4 + 6\sqrt{y} - 10$

(ii)　　$g(x, y, z) = \dfrac{x\sin(y)}{z^2}$

Solution: The first order partial derivative is evaluated as follows.

(i) $f(x, y) = x^4 + 6\sqrt{y} - 10$

We first evaluate the partial derivative of the function $f(x, y) = x^4 + 6\sqrt{y} - 10$ with regard to x considering y as a constant. Therefore, the partial derivative with regard to 'x' is,

$$f_x(x, y) = 4x^3$$

In this case, the second and the third terms differentiate to '0' (zero). Since as per the standard rule of differentiation, the term that is considered as a constant differentiates to zero. Now when we differentiate with respect to 'x' and consider 'y' as a constant, then all the terms with 'y' will be considered as constants and therefore will be differentiated to '0' (zero).

Accordingly, when we differentiate with respect to 'y' and consider 'x' as a constant, then the term which includes 'x' will be considered as constants and therefore will be differentiated to '0' (zero). Following partial derivative is evaluated with regard to 'y':

$$f_y(x, y) = \dfrac{3}{\sqrt{y}}$$

Therefore, the partial derivative with regard to x is,　$f_x(x, y) = 4x^3$

And the partial derivative with regard to 'y' is,　$f_y(x, y) = \dfrac{3}{\sqrt{y}}$

(ii) $g(x, y, z) = \dfrac{x\sin(y)}{z^2}$

We evaluate the derivatives with reference to 'x' and 'y' of the given function $g(x, y, z) = \dfrac{x\sin(y)}{z^2}$ when 'z' is considered as a constant. On the right-hand side, the 'z' in the denominator is a constant.

The derivatives for x and y are as follows,

$$g_x\,(x, y, z) = \frac{\sin(y)}{z^2}$$

$$g_y\,(x, y, z) = \frac{x\cos(y)}{z^2}$$

Additionally, to differentiate 'x' and 'y' with regard to 'z', the derivative is,

Example 4. Find dy/dx for $3y^4 + x^7 = 5x$.

Solution: To find dy/dx for $3y^4 + x^7 = 5x$ we first consider y as a function of x, or we can say that $y = y(x)$.

When a term including y is differentiated with regard to x then dy/dx is added to that term as illustrated below.

On either side we differentiate with regard to x as follows:

$$12y^3\frac{dy}{dx} + 7x^6 = 5$$

Solving for dy/dx,

$$\frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}$$

Now we will discuss about the **second partial derivative** of 'f'. It is represented in the following four notations.

Notations

(i) Differentiate 'f' two times with regard to 'x', i.e., we first differentiate 'f' with reference to 'x' and then again differentiate the result with reference to 'x' that is obtained after the first differentiation.

Thus,　　　$\partial^2 f\,/\,\partial x^2$　or　f_{xx}

(ii) Differentiate 'f' two times with regard to 'y', i.e., we first differentiate 'f' with reference to 'y' and then again differentiate the result with reference to 'y' that is obtained after the first differentiation.

Thus,　　　$\partial^2 f\,/\,\partial y^2$　or　f_{yy}

Mixed Partials

(iii) We first differentiate 'f' with reference to 'x' and then again differentiate the result with reference to 'y' that is obtained after the first differentiation.

Thus,　　　$\partial^2 f\,/\,\partial y\,\partial x$ or　f_{xy}

(iv) We first differentiate 'f' with reference to 'y' and then again differentiate the result with reference to 'x' that is obtained after the first differentiation.

Thus, $\partial^2 f / \partial x\, \partial y$ or $\quad \boldsymbol{f_{yx}}$

Example 5. Given is $f(x, y) = 3x^2y + 5x - 2y^2 + 1$, then find the partial derivatives $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ and f_{yx}.

Solution: We evaluate the partial derivatives in the following way.

We first differentiate f with regard to x when y is considered as a constant. This will yield,

$$f_x = 6xy + 5$$

Next, we first differentiate f with regard to y when x is considered as a constant. This will yield,

$$f_y = 3x^2 - 4y$$

The f_{xx} is considered as the second partial derivative of f_x, i.e., the partial derivative of f_x with regard to x.

This will yield,

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(6xy + 5) = 6y$$

The f_{yy} is considered as the second partial derivative of f_y, i.e., the partial derivative of f_y with regard to y.

This will yield,

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(3x^2 - 4y) = -4$$

The f_{xy} is considered as the mixed second partial derivative of f_x with regard to y, i.e., the partial derivative of f_x with regard to y.

This will yield,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(6xy + 5) = 6x$$

The f_{yx} is considered as the mixed second partial derivative of f_y with regard to x, i.e., the partial derivative of f_y with regard to x.

This will yield,

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(3x^2 - 4y) = 6x$$

Consequently, see the result in case of the mixed second partial derivatives, f_{xy} and f_{yx}, both are similar as both the result yield '$6x$', i.e.,

$$\boldsymbol{f_{xy} = f_{yx}}$$

Example 6. Given is $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the partial derivaties of first order and second order.

Solution: The partial derivaties of first order and second order are evaluated as follows.

$$\frac{\partial z}{\partial x} = 8x - 8y^4$$

$$\frac{\partial z}{\partial y} = -8x(4y^3) + 35y^4 = -32xy^3 + 35y^4$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 8$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(-32xy^3 + 35y^4) = -32x(3y^2) + 140y^3$$

$$= -96xy^2 + 140y^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-32xy^3 + 35y^4) = -32y^3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x - 8y^4) = -32y^3$$

Check Your Progress

1. What is a partial differential equation?
2. What is power rule?
3. Find the derivative of $5x^3$ using the power rule.

3.3 HOMOGENEOUS FUNCTIONS

A **homogeneous function** of two variables 'x' and 'y' is stated as a real-valued function which satisfies and fulfills the condition,

$$f(\alpha x, \alpha y) = \alpha^k f(x, y)$$

for specific constant 'k' and all real numbers 'α'. The constant k is termed as the **degree of homogeneity**.

Definition. A function $f(x, y)$ is said to be a **homogeneous function** of **degree** 'n', if,

$$f(x, y) = x^n f\left(\frac{y}{x}\right)$$

Where 'n' is a **real number**.

Definition. If 'f' be a function of 'n' variables defined on a set 'S' for which $(tx_1, \ldots, tx_n) \in S$ whenever $t > 0$ and $(x_1, \ldots, x_n) \in S$. Then 'f' is termed as the **homogeneous function** of **degree** 'k' if,

$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in S$ and $t > 0$

Since the definition includes or defines the relationship or association between the specified value of the function at $(x_1, \ldots, x_n)$ and its precise values at the points $(tx_1, \ldots, tx_n)$ in which 't' can be some positive number. Remember that this is restricted to all those particular functions where $(tx_1, \ldots, tx_n)$ is every time evaluated in the domain when $t > 0$ and $(x_1, \ldots, x_n)$ exists in the domain.

Certain specific domains which possess this property includes the set of all real numbers, the **set** of **positive real numbers**, the **set** of **non-negative real numbers**, the **set** of all n-**tuples** $(x_1, \ldots, x_n)$ of **real numbers**, the **set** of n-**tuples** of **positive real numbers** and the **set** of n-**tuples** of **non-negative real numbers**.

A **function** is said to be '**homogeneous of degree k**' if its every single argument is multiplied with some number $t > 0$, then consequently the value of the function too is multiplied with t^k. For instance, a function is considered as '**homogeneous of degree 1**' when its all or whole arguments are multiplied with some number $t > 0$, and accordingly the value of the function too is multiplied with the identical or similar number t.

Definition. A function of two variables 'x' and 'y' is of the form,

$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \ldots + a_{n-1} xy^{n-1} + a_n y^n$

Here each term is of **degree** 'n' and is termed as the **homogeneous function** or when it is represented or expressed in the form,

$y^n g(x/y)$ or $x^n g(y/x)$

Then, for example,

$f(x, y) = x^2 + y^2 / x + y$

This is termed as the **homogeneous function** of '**degree 1**'.

3.3.1 Can the Sum of Two Homogeneous Functions Be Essentially Homogeneous?

Absolutely not, the sum of two homogeneous functions cannot be essentially homogeneous.

Let us prove it with the help of the following example.

Consider that $h(x) = 1 + x$.

Where the constant function $f(x) = 1$ is defined as the homogeneous of degree 0 while the function $g(x) = x$ is defined as the homogeneous of degree 1.

However, h is by no means homogeneous of any degree.

If h is considered as or exists as homogeneous of degree k, then for all t and all x there will be $1 + tx = t^k(1 + x)$, which specifically implies that $1 + 2x = 2^k (1 + x)$ (for $t = 2$), which sequentially implies that $1 = 2^k$ (for $x = 0$) and $3 = 2(2^k)$, which are varying or inconsistent.

Example 7. Check the following given functions and determine whether each is homogeneous or not. If homogeneous then of what degree?

  (i) $3x + 4y$

  (ii) $3x + 4y - 2$

  (iii) $2x^2 + xy$

  (iv) $x^2 + x^3$

Solution: The given functions are determined for homogeneous as follows.

  (i) $3x + 4y$

The function $3x + 4y$ is Homogeneous of Degree 1, because,

$$3(tx) + 4(ty) = t(3x + 4y)$$

  (ii) $3x + 4y - 2$

This function is 'Not' Homogeneous.

Assume that for all t and all (x, y) there exists certain specific value of k such that,

$$3(tx) + 4(ty) - 2 = t^k (3x + 4y - 2)$$

Then precisely, for all (x, y) when $t = 2$,

$$6x + 8y - 2 = 2^k (3x + 4y - 2)$$

Therefore, taking $(x, y) = (1, 0)$ we have,

$$6 - 2 = 2^k (3 - 2) \text{ or } \quad 2^k = 4 \qquad \qquad ….(1)$$

And subsequently, taking $(x, y) = (0, 1)$ we have,

$$8 - 2 = 2^k (4 - 2) \text{ or } \quad 2^k = 3 \qquad \qquad ….(2)$$

The two conditions that outcome in Equations (1) and (2) are inconsistent or varying, therefore the function $3x + 4y - 2$ is not at all homogeneous of any degree.

  (iii) $2x^2 + xy$

The function $2x^2 + xy$ is Homogeneous of Degree 2, because,

$$2 (tx)^2 + (tx)(ty) = t^2 (2x^2 + xy)$$

(iv) $x^2 + x^3$

This function is 'Not' Homogeneous.

Assume that for all t and all x there exists certain specific value of k such that,

$(tx)^2 + (tx)^3 = t^k (x^2 + x^3)$

Then precisely, for all x when $t = 2$,

$4x^2 + 8x^3 = 2^k (x^2 + x^3)$

Therefore, taking $x = 1$ we have,

$6 = 2^k$ ....(3)

And subsequently, taking $x = 2$ we have,

$20/3 = 2^k$ ....(4)

The two conditions that outcome in Equations (3) and (4) are inconsistent or varying, therefore the function $x^2 + x^3$ is not at all homogeneous of any degree.

3.3.2 Partial Derivatives of Homogeneous Functions

The partial derivative of homogeneous functions can be explained with the help of the following proposition or theorem.

Theorem. Considering that 'f' be a differentiable function of 'n' variables that is specifically homogeneous of degree 'k', then subsequently each of its partial derivatives f'_i for $(i = 1, \ldots, n)$ is Homogeneous of Degree '$k - 1$'.

Proof.

The proof of the theorem comprises the homogeneity of 'f' such that, for all $(x_1, \ldots, x_n)$ and for all $t > 0$.

$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$

Differentiating both sides with regard to x_i, i.e., the left-hand side and the right-hand side of this equation we have,

$tf'_i (tx_1, \ldots, tx_n) = t^k f'_i (x_1, \ldots, x_n)$

Now both the sides of the equation are divided by 't' and we have,

$f'_i (tx_1, \ldots, tx_n) = t^{k-1} f'_i (x_1, \ldots, x_n)$

Hence, proved that f'_i is homogeneous of degree '$k - 1$'.

3.3.3 Euler's Theorem for Homogeneous Functions

The theory that a homogeneous function of certain specific degree exists was first discovered by **Leonhard Euler**.

Definition. A function $f(x, y)$ is a homogeneous function of 'x' and 'y' of degree 'n' if,

$f(tx, ty) = t^n f(x, y)$ for $t > 0$

Euler's Theorem for Homogeneous 1. If 'z' is a homogeneous function of 'x' and 'y' of degree 'n' and first order partial derivative of 'z' exists, then,

$$xz_y + yz_x = nz$$

Consider that $f(x, y)$ is a homogeneous function of 'x' and 'y' of order or degree 'n' so as to,

$$f(tx, ty) = t^n f(x, y)$$

Now define $x' \equiv xt$ and $y' \equiv yt$, such that,

$$nt^{n-1} f(x, y) = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial t}$$

$$= x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'}$$

$$= x \frac{\partial f}{\partial (x\,t)} + y \frac{\partial f}{\partial (y\,t)}.$$

Taking $t = 1$, we have,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y).$$

Generalizing this to an arbitrary number of variables, we have,

$$x_i \frac{\partial f}{\partial x_i} = n f(\mathbf{x}),$$

Definition. The function '$F: \mathbb{R}^n \to \mathbb{R}$' is defined as homogeneous of Degree 'k' when for all 'λ',

$$F(\lambda x) = \lambda^k F(x)$$

Remember that the homogeneity of 'Degree 1' is weaker as compared to linearity. All the linear functions are considered as homogeneous of 'Degree 1' however not conversely, such as the function $f(x, y)$ is defined as homogeneous of degree 1 though not linear.

Euler's Theorem for Homogeneous 2. If '$F: \mathbb{R}^n \to \mathbb{R}$' is differentiable at '$x$' and homogeneous of Degree 'k', then

$$\nabla F(x) \cdot x = kF(x)$$

Proof. Let 'x' is fixed. Now consider that the function,

$$H(\lambda) = F \circ G(\lambda)$$

Here $G : \mathbb{R} \to \mathbb{R}^n$, such that,

$$G(\lambda) = \lambda x$$

According to the Chain Rule,

$DH(\lambda) = DF(G(\lambda))\, DG(\lambda)$

Taking $\lambda = 1$, when this equation is evaluated, then we have,

$$DH(1) = \nabla F(x) \cdot x \qquad \qquad \ldots(5)$$

Alternatively, we can define that as per the homogeneity,

$$H(\lambda) = \lambda^k x$$

Now when we differentiate the right-hand side of the equation then it will yield,

$$DH(\lambda) = k\lambda^{k-1} F(\lambda x)$$

Taking $\lambda = 1$, when this equation is evaluated, then we have,

$$DH(1) = kF(x) \qquad \qquad \ldots(6)$$

When we combine Equations (5) and (6) then it will yield the theorem, i.e.,

$$\nabla F(x) \cdot x = kF(x)$$

Chain Rule

In calculus, the **chain rule** formula is precisely used to compute the derivatives of two or more functions. Specifically, for the functions *f* and *g*, the chain rule states that the derivatives *f* o *g*, the function that defines *x* to *f(g(x))* as the derivatives of *f* and *g*, and the product of functions as shown below.

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Equivalently, this can be stated using variables. Consider that for all '*x*',

$$F = f' \circ g$$

Or, $\qquad F(x) = f(g(x))$

This can also be written as,

$$F'(x) = f'(g(x))\, g(x)$$

In **Leibniz Notation,** the **Chain Rule** can be written in the following form. If '*z*' is considered as a vaiable that depends of variable '*y*', and where variable '*y*' is in turn dependent on variable '*x*' such that both '*y*' and '*z*' are consequently be the dependent variables. Then the variable '*z*' too will depend on variable '*x*' through variable '*y*'. The chain rule then can be stated as,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

The above derived two forms of the chain rule are interrelated and represented as follows, when $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x) = f'(g(x))g'(x)$$

3.4.1 Verification of Euler's Theorem

Thus, as per Euler Theorem, "If 'u' is a homogeneous function of 'x' and 'y' of degree 'n', then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Now we will prove the theorem and verify the result.

Proof of Euler's Theorem:

Since as per Euler Theorem stated above, 'u' is defined as the homogeneous function of 'x' and 'y' of degree 'n', hence 'u' can also be represented in the following form:

$$u = x^n f\left(\frac{y}{x}\right) \qquad \dots(8)$$

Now when we partially differentiate Equation (8) with regard to 'x', we get the following form of equation,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^n f\left(\frac{y}{x}\right)\right]$$

$$= \left[f\left(\frac{y}{x}\right)\right]nx^{n-1} + x^n\left[f'\left(\frac{y}{x}\right)\right]\left(-\frac{y}{x^2}\right).$$

Therefore,

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1}y.f'\left(\frac{y}{x}\right) \qquad \dots(9)$$

Now when we partially differentiate Equation (9) with regard to 'y', we get the following form of equation,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^n f\left(\frac{y}{x}\right)\right] = x^n\left[f'\left(\frac{y}{x}\right)\right].\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

Therefore,

$$y\frac{\partial u}{\partial y} = y.x^{n-1}f'\left(\frac{y}{x}\right). \qquad \dots(10)$$

Now on adding Equations (9) and (10) we get the following form of equation:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu \qquad \text{(From Equation 8))}$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Hence proved and verified

Remember that it is possible to **extend** the **'Euler's theorem'** to the **'homogeneous function'** of **'any number of variables'**.

Accordingly, if a function $f(x_1, x_2, x_3, x_4, \ldots, x_n)$ is a homogeneous function of $x_1, x_2, x_3, x_4, \ldots, x_n$ of Degree 'n', then we get the equation of the form,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_n \frac{\partial f}{\partial x_n} = nf$$

Example 8. Find the degree of homogeneity and then verify the Euler theorem for the given equation of the form,

$$u = x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4$$

Solution: Given is,

$$u = x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4$$

Evidently, in this equation 'u' is the homogeneous function for 'x' and 'y', and is of degree '4'.

Therefore, as per the Euler theorem we require the equation of the following given form.

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 4u.$$

Now we will verify the same as follows.

We deduce the equation to obtain,

$$\left(\frac{\partial u}{\partial x}\right) = 4x^3 - 9x^2y + 10xy^2 + 4y^3,$$

And,

$$\left(\frac{\partial u}{\partial y}\right) = -3x^3 + 10x^2y + 12xy^2 - 8y^3$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x(4x^3 - 9x^2y + 10xy^2 + 4y^3)$$

$$+ y(-3x^3 + 10x^2y + 12xy^2 - 8y^3)$$

Hence, we obtain the equation,

$$= 4(x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4)$$

$$= 4u$$

Because, given is that $u = x^4 - 3x^3y + 5x^2y^2 + 4xy^3 - 2y^4$.

Henceforth, this verifies the legitimacy of the Euler's theorem.

3.5 MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE AND TWO VARIABLES

By differentiation we can find the maxima and minima for any function of one variable $f(x)$ and two variables $f(x, y)$.

3.5.1 Function of One Variable

The maxima and minima for any function of one variable **$f(x)$** takes place when,

$f'(x) = 0$

Descriptions

Therefore,

1. $x = a$ is **maximum** when $f'(a) = 0$ and $f''(a) < 0$

2. $x = a$ is **minimum** when $f'(a) = 0$ and $f''(a) > 0$

The point at which **$f''(a) = 0$** and **$f'''(a) \neq 0$** is termed as **'point of inflection'**. As per geometry, a curve that is in the two-dimensional plane **(x, y)** can be represented by the equation **$y = f(x)$**. The graph of the curve is represented by the **function $f(x)$**.

Definitions

1. A **function $f(x)$** is defined to be **maximum** at $x = a$, when there exists a **positive number 'δ'** such that for all values of 'h' in the interval **$(-\delta, \delta)$** excluding **'0'** we have,

   $f(a + h) < f(a)$

2. A **function $f(x)$** is defined to be **minimum** at $x = a$, when there exists a **positive number 'δ'** such that for all values of 'h' in the interval **$(-\delta, \delta)$** excluding **'0'** we have,

   $f(a + h) > f(a)$

These **maximum value** and **minimum value** of a function is also termed as the **'extreme values'** and the **points** at which these specific points occur are termed as the **'points of maxima** and **minima'**.

Additionally, these points of maxima and minima of a function showing the extreme values are termed as **'turning points'**.

Properties of Maxima and Minima for a Function in One Variable

Following are the properties of the maxima and minima of a function showing the extreme values.

1. Any function '$y = f(x)$' is defined as maximum at '$x = a$', when the sign of dy/dx is changed from positive to negative at the moment 'x' passes through 'a'.

2. Any function '$y = f(x)$' is defined as minimum at '$x = a$', when the sign of dy/dx is changed from negative to positive at the moment 'x' passes through 'a'.

3. Same function may have numerous maximum or minimum values.

4. In between the two equal/equivalent values of a function there must be at least one maximum value and one minimum value.

5. If there is no change in the sign of dy/dx, i.e., from negative to positive or positive to negative, when 'x' passes through 'a', then at the specified '$x = a$' we can define that 'y' is neither maximum nor it is minimum.

6. The 'Maximum' value and the 'Minimum' value of a function essentially occurs in consecutive or alternate sequence.

Necessary Condition for Maximum and Minimum Values

Definition. A necessary or essential condition required for $f(x)$ to be either maximum or minimum at '$x = a$' is '$f'(a) = 0$'.

Stationary Values

Definition. A function $f(x)$ is assumed to be stationary at '$x = a$' when '$f'(a) = 0$'.

Therefore, any function $f(x)$ can be maximum or minimum at '$x = a$' when it is stationary at '$x = a$'.

Appropriate Conditions for Maximum and Minimum Values

Following are the sufficient or appropriate conditions that are required for the function for maximum and minimum values.

1. The **function $f(x)$** will be **maximum** at **'$x = a$'** when **'$f'(a) = 0$'** and **'$f''(a)$'** is **'negative'**.

2. The **function $f(x)$** will be **minimum** at **'$x = a$'** when **'$f'(a) = 0$'** and **'$f''(a)$'** is **'positive'**.

As a general rule, when,

$$f'(a) = f''(a) = f'''(a) = \ldots \ldots = f^{(n-1)}(a) = 0$$

And,

$$f^{(n)}(a) \neq 0$$

Then, for maximum and minimum values the '**n**' is considered as an even integer. Additionally, for '$f^{(n)}(a)$' to be maximum it has to be negative while for '$f^{(n)}(a)$' to be minimum it has to be positive.

3.5.2 Function of Two Variables

The function of two variables can be written using the given form of equation,

$$z = f(x, y)$$

In this equation, the variables '**x**' and '**y**' are considered as the **independent variables** while the variable '**z**' is defined as the **dependent variable**. The graph of the function of two variables is represented as a surface for a three-dimensional space.

For Example

$$z = \frac{1}{1 + x^2 + y^2}.$$

Here, 'z' represents the surface height above a specified point (x, y) in the '$x - y$' plane.

Now, the graph for the function $z = f(x, y)$ can have either **maximum points** or **minimum points** or **both**.

Consequently, a point (a, b) that is considered either as a **maximum point** or as a **minimum point** or as a **saddle point** is termed as '**stationary point**' or '**critical point**'. The critical points or saddle points for the functions of a single variable can be defined as the values of the function when the derivative is either equal to '**zero**' or it '**does not exist**' at all. Principally, in the same way we can define the functions of two or more variables.

Definition Stationary or Critical Point

Consider that $z = f(x, y)$ is a function of two variables specifically defined for an open set that contains the point (x_0, y_0). This **point (x_0, y_0)** is termed as the **critical point** of a function of two variables 'f' when it holds any of the given two conditions, i.e.,

1. Either $\qquad f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

This can equivalently be expressed as, $\nabla f(x_0, y_0) = \vec{0}$

2. Or $\qquad f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ does not exist at all.

Definitions Maximum Points or Minimum Points

1. A **function $f(x, y)$** can be considered **minimum** at the **point (a, b)** when for all the points (x, y) in any specific region about (a, b) we have,

$$f(x, y) \geq f(a, b)$$

2. A **function** $f(x, y)$ can be considered **maximum** at the **point (a, b)** when for all the points (x, y) in any specific region about (a, b) we have,

$f(x, y) \leq f(a, b)$

3. Consider that $f(x, y)$ is a **function** of **two independent variables 'x'** and 'y' which is assumed to be continuous for all values of these variables 'x' and 'y' in the neighbouring region of their values 'a' and 'b', respectively.

Subsequently, $f(a, b)$ can have **maximum value** or **minimum value** of the **function $f(x, y)$** provided that $f((a + h), (b + k))$ is either *less than* or *greater than $f(a, b)$,* **positive** or **negative**, for almost all the appropriate and small independent values of 'h' and 'k', so long as both are **not equal to '0'**.

Necessary Conditions for the Existence of Maximum or Minimum of Two Variables

The necessary conditions that must exist for the maximum value or minimum value of two variables, $f(x, y)$ at '$x = a$' and '$y = b$' when the following given expression,

$$f((a + h), (b + k)) - f(a, b)$$

is of invariable or consistent sign, positive or negative, for almost all the appropriate and small independent values of '**h**' and '**k**', so long as both are **not equal to '0'**.

The function $f(x, y)$ at '$x = a$' and '$y = b$' will be maximum when the sign of the $f((a + h), (b + k)) - f(a, b)$ is negative. Alternately, the function $f(x, y)$ at '$x = a$' and '$y = b$' will be minimum when the sign of the $f((a + h), (b + k)) - f(a, b)$ is positive.

Using the Taylor's Theorem given for the function of two variables, we get the following equation.

$f((a + h), (b + k))$

$$= f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}}$$

$$+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \right)_{\substack{x=a \\ y=b}} + \ldots$$

$f((a + h), (b + k)) - f(a, b)$

$$= h\left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} + k\left(\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}} + \text{Expressions of the second order and above}$$

in 'h' and 'k'

Therefore, the necessary conditions state that the function $f(x, y)$ must have maximum or minimum at '$x = a$' and '$y = b$' such that,

$$\left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} = 0$$

And,

$$\left(\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}} = 0$$

These conditions are termed as necessary for maxima and minima to exist but are not sufficient or appropriate.

Example 9. Given is $dy/dx = x(x - 1)^2 (x - 3)^3$, then find the maximum value and minimum value of 'y'.

Solution: To obtain the maximum value and minimum value of 'y',

$$dy/dx = 0$$

Specifically, $x(x - 1)^2 (x - 3)^3 = 0$

Or else, $\quad\quad\quad x = 0, x = 1, x = 3$

For 'y' at '$x = 0$'

The dy/dx is defined as positive if 'x' is somewhat less than '0'. Alternatively, the dy/dx is defined as negative if 'x' is somewhat greater than '0'. Therefore, the sign of dy/dx will change from positive to negative when 'x' will pass through the value '0'. Consequently, we can state that when '$x = 0$' then 'y' is maximum.

For 'y' at '$x = 1$'

The dy/dx is defined as negative if 'x' is somewhat less than '1'. Alternatively, the dy/dx is defined as negative again if 'x' is somewhat greater than '1'. Therefore, the sign of dy/dx will 'NOT' change at all when 'x' will pass through the value '1'. Consequently, we can state that when '$x = 1$' then 'y' is neither maximum nor minimum.

For 'y' at '$x = 3$'

The dy/dx is defined as negative if 'x' is somewhat less than '3'. Alternatively, the dy/dx is defined as positive if 'x' is somewhat greater than '3'. Therefore, the sign of dy/dx will change from negative to positive when 'x' will pass through the value '3'. Consequently, we can state that when '$x = 3$' then 'y' is minimum.

Check Your Progress

6. When is a function $f(x)$ assumed to be stationary?

7. What are the critical points?

8. What is the definition of maximum points for a function of two variables?

3.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A differential equation which expresses one or more measures in terms of partial derivatives is called a partial differential equation.

2. As per the Power Rule, the derivative of x^n is $nx^{(n-1)}$.

3. $15x^2$.

4. A function is considered as 'homogeneous of degree 1' when its all or whole arguments are multiplied with some number $t > 0$, and accordingly the value of the function too is multiplied with the identical or similar number t.

5. For the functions f and g, the chain rule states that the derivatives $f'' g$, the function that defines x to $f(g(x))$ as the derivatives of f and g, and the product of functions as shown below.

   $(f \circ g)' = (f' \circ g) \cdot g'$

6. A function $f(x)$ is assumed to be stationary at '$x = a$' when '$f'(a) = 0$'.

7. The critical points or saddle points for the functions of a single variable can be defined as the values of the function when the derivative is either equal to 'zero' or it 'does not exist' at all.

8. A function $f(x, y)$ can be considered maximum at the point (a, b) when for all the points (x, y) in any specific region about (a, b) we have, $f(x, y) \leq f(a, b)$.

3.7 SUMMARY

- A partial derivative of a function of several variables is its derivative with respect to one of those variables, while the remaining others are considered as constant.

- A differential equation which expresses one or more measures in terms of partial derivatives is called a partial differential equation.

- Notations for second partial derivative of 'f'.

  (i) Differentiate 'f' two times with regard to 'x', thus $\partial^2 f / \partial x^2$ or f_{xx}

  (ii) Differentiate 'f' two times with regard to 'y', thus, $\partial^2 f / \partial y^2$ or f_{yy}

- For mixed partials, we first differentiate '*f*' with reference to '*x*' and then again differentiate the result with reference to '*y*' that is obtained after the first differentiation. Thus, $\partial^2 f / \partial y \partial x$ or f_{xy}. Now differentiate '*f*' with reference to '*y*' and then again differentiate the result with reference to '*x*' that is obtained after the first differentiation. Thus, $\partial^2 f / \partial x \partial y$ or f_{yx}

- A homogeneous function of two variables '*x*' and '*y*' is stated as a real-valued function which satisfies and fulfills the condition, $f(\alpha x, \alpha y) = \alpha^k f(x, y)$ for specific constant '*k*' and all real numbers 'α'. The constant *k* is termed as the degree of homogeneity.

- A function $f(x, y)$ is a homogeneous function of '*x*' and '*y*' of degree '*n*' if, $f(tx, ty) = t^n f(x, y)$ for $t > 0$.

- Euler's Theorem for Homogeneous 1. If '*z*' is a homogeneous function of '*x*' and '*y*' of degree '*n*' and first order partial derivative of '*z*' exists, then, $xz_y + yz_x = nz$.

- Euler's Theorem for Homogeneous. State that, "If '$F: \mathbb{R}^n \to \mathbb{R}$' is differentiable at '*x*' and homogeneous of Degree '*k*', then $\nabla F(x) \cdot x = kF(x)$".

- A function $f(x)$ is defined to be maximum at $x = a$, when there exists a positive number 'δ' such that for all values of '*h*' in the interval $(-\delta,)$ excluding '0' we have, $\qquad f(a + h) < f(a)$

- A function $f(x)$ is defined to be minimum at $x = a$, when there exists a positive number 'δ' such that for all values of '*h*' in the interval $(-\delta, \delta)$ excluding '0' we have, $\qquad f(a + h) > f(a)$

- A function $f(x, y)$ can be considered minimum at the point (a, b) when for all the points (x, y) in any specific region about (a, b) we have, $f(x, y) \geq f(a, b)$.

- A function $f(x, y)$ can be considered maximum at the point (a, b) when for all the points (x, y) in any specific region about (a, b) we have, $f(x, y) \leq f(a, b)$.

3.8 KEYWORDS

- **Derivative:** Derivative, in mathematics, the rate of change of a function with respect to a variable.

- **Maxima and minima:** In mathematical analysis, the maxima and minima of a function, known collectively as extrema, are the largest and smallest value of the function, either within a given range or on the entire domain of a function.

- **Homogeneous function:** A homogeneous function is one with multiplicative scaling behaviour: if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor.

• **Constant:** A quantity or parameter that does not change its value whatever the value of the variables, under a given set of conditions.

3.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. For the following given functions, find all the first order partial derivatives.

   $6x^3 + \sqrt{y} - 1$.

2. Determine whether the function $4x^2 + y^2 - 1$ is homogeneous or not. If homogeneous then of what degree?

3. Discuss properties of maxima and minima for a function in one variable.

4. Given is $f(x, y) = x^2y + 7xy - 8$, then find the partial derivatives f_{xy} and f_{yx}.

Long Answer Questions

1. Prove that the sum of two homogeneous functions cannot be essentially homogeneous.

2. State and prove Euler's theorem.

3. Find the degree of homogeneity and then verify the Euler theorem for the given equation of the form, $u = x^4 - x^3y - 9x^2y^2 + 4xy^3 - y^4$

4. Discuss the necessary conditions for the existence of maximum or minimum of two variables.

5. Given is $dy/dx = 2x(x - 4)^2 (x - 5)^3$, then find the maximum value and minimum value of 'y'.

3.10 FURTHER READINGS

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Sharma, A.K. 2004. *Text book of Differential Calculus*. New Delhi: Discovery publishing house.

Narayanan, S, T.K. Manickavasagam Pillai. 2009. *Differential Equations and its applications*. Chennai: S.Viswanathan(Printers & Publishers) Pvt. Ltd.

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UNIT 4   POLAR COORDINATES AND ASYMPTOTES

Structure

4.0   INTRODUCTION

In this unit, you will learn about polar coordinates and concept of curvature in polar coordinates. In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

The actual term polar coordinates has been attributed to Gregorio Fontana and was used by 18th-century Italian writers. The term appeared in English in George Peacock's 1816 translation of Lacroix's Differential and Integral Calculus. Alexis Clairaut was the first to think of polar coordinates in three dimensions, and Leonhard Euler was the first to actually develop them. Polar coordinates are used often in navigation as the destination or direction of travel can be given as an angle and distance from the object being considered.

Further in this unit, you will earn about asymptotes and its types. Asymptotes convey information about the behavior of curves in the large, and determining the asymptotes of a function is an important step in sketching its graph.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of polar coordinates
- Know the significance of radius of curvature in polar coordinates
- Discuss asymptotes and its types
- Describe method of finding asymptotes

4.2 POLAR COORDINATES

Mathematically, the **polar coordinate** method can be defined as a two-dimensional coordinate system wherein every single point on a plane is precisely established or determined through a distance with regard to **'a reference point and an angle from a reference direction'**.

The reference point is termed as the **'pole'** and is considered as analogous or equivalent to the **Cartesian coordinate** or Rectangular or '*x-y*' system. The ray that originates from the pole towards the reference direction is termed as the **'polar axis'**. The distance from the pole is termed as the **'radial coordinate'** or **'radius'** and the **'angle'** is termed as the **'angular coordinate'**, **'polar angle'** or **'azimuth'**.

Now using the Cartesian coordinate system, at the particular point for the given or specified coordinates (x, y) to define the point we start at the origin and then subsequently '*x*' is moved horizontally while '*y*' vertically, as shown in the Figure 4.1.
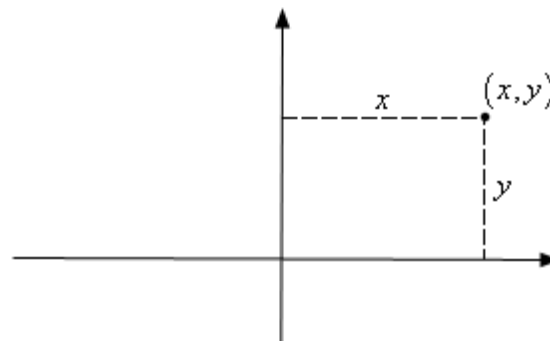
Fig. 4.1 *Coordinates 'x' and 'y'*

A point in a two-dimensional space can be determined on the basis of the angle 'θ' made on the **positive '*x*-axis'**, as shown in Figure 4.2. The coordinates of the point are defined as the distance of the specified point taken from the origin '*r*' and the extent or degree required in rotating on the positive '*x*-axis', i.e., 'θ' are termed as the **polar coordinates (r, θ)**.
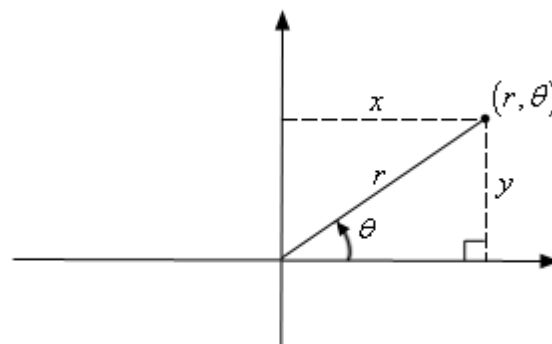
Fig. 4.2 *Polar Coordinates (r, θ)*

The **angle θ** is measured as **positive** in the **counterclockwise** direction from the **polar axis** while the **angle θ** is measured as **negative** in the **clockwise** direction from the **polar axis**.

Consider that '*r*' is any number. For plotting a point '*P*' to relate with '*r*' and 'θ',

1. When '*r*' is positive then '*P*' is defined as the intersection of the circle with radius '$|r|$' assuming that pole is the centre of the circle and also the angle 'θ' that is formed by the ray from the pole.

2. When '*r*' is '0' then '*P*' is defined at the pole irrespective of whatsoever the angle 'θ' measures.

3. When '*r*' is negative then '*P*' is defined from the pole at some distance '*r*' on the ray which is exactly opposite to the ray that forms angle 'θ', i.e., on the ray of angle 'θ + π'.

In all of the above defined three conditions, '*P*' is represented by (r, θ) where the '*r*' and 'θ' are termed as the polar coordinates of '*P*', when the point (r, θ) is positioned on the circle with centre at the pole and radius as '$|r|$'.

Consequently, the pole is considered as the midpoint or center for the points (r, θ) and also $(-r, \theta)$.

Remember that we can consider the point $(-r, \theta + \pi)$ equivalent as the point (r, θ). Additionally, when the angle is changed by '2π' then there will be no change in the point, i.e.,

$$(r, \theta) = (r, \theta + 2\pi) = (r, \theta + 4\pi) = \ldots \ldots = (r, \theta + 2n\pi)$$

Where '*n*' can be any integer, positive or negative.

Therefore, in the plane we can define the polar coordinates (r, θ) through,

'*r*'　　　　　　　= Distance from the Origin or Radial Coordinate

'θ'　　　　　　　= Polar Angle or Angular Coordinate

'θ ∈ [0, 2π)'　　= Counterclockwise Angle

Therefore, as per the Figure 4.3, we can make the convention that,

$$(-r, \theta) = (r, \theta + \pi) \text{ or } (r, \theta \pm \pi)$$
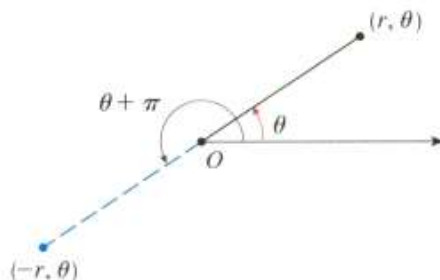
Fig. 4.3 *Polar Coordinates*

The Polar coordinates (r, θ) can also be stated in terms of Cartesian coordinates as follows (Refer Figure 4.4).

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r^2 = x^2 + y^2 \qquad \text{or} \qquad r = \sqrt{x^2 + y^2}$$

$$\tan \theta = y/x \qquad \text{or} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Here 'r' is the termed as the radial distance from the origin while 'θ' is termed as the counterclockwise angle on the *x*-**axis**.
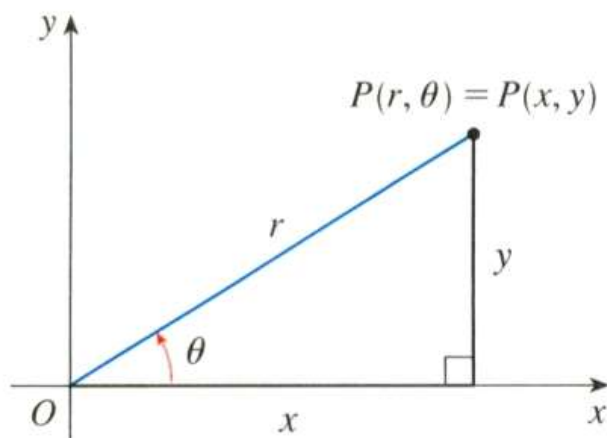
Fig. 4.4 *Polar Coordinate to Cartesian Coordinate*

The form $\tan^{-1}\left(\frac{y}{x}\right)$. can be interpreted by means of 'two-argument inverse tangent' that takes the sign of 'x' and 'y' for determining the quadrant of 'θ'.

How the Polar Coordinates are Converted to Cartesian Coordinates

For, $\qquad x = r \cos \theta \qquad$ and $\quad y = r \sin \theta$

We can convert from Polar form to Cartesian as follows.

$$x^2 + y^2 = (r \cos \theta)^2 + (r \sin \theta)^2$$

$$= r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

Hence, $\qquad r^2 = x^2 + y^2$

The above formula is very significant. Now on taking the square root of left-hand side and right-hand side we have,

$$r = \sqrt{x^2 + y^2}$$

Mathematically, there must be a plus or a minus sign in front of the root because '*r*' is either positive or negative. Taking '*r*' as positive we have,

$$y/x = r \sin \theta / r \cos \theta = \tan \theta$$

On both sides taking inverse tangent we have,

$$\theta = \tan^{-1}(y/x)$$

Remember that the inverse tangent will only yield values in the following given range,

$$(-\pi/2) < \theta < (\pi/2)$$

The another possible angle is '$\theta + \pi$'.

The point (r, θ) can be expressed as an ordered pair (r, θ) and is termed as 'polar notation', the equation for the curve that is represented as polar coordinates is termed as 'polar equation' while the curve that is plotted in polar coordinates is termed as 'polar plot'. Similarly, we can plot the Cartesian curves on the rectilinear axes while we can draw the polar plots on the radial axes as shown in Figure 4.5. Normally, in polar notation the angles are expressed either in degrees or in radians where 2π rad is equal to $360°$. Figure 4.5 displays a polar grid with different angles that are labelled in degrees.
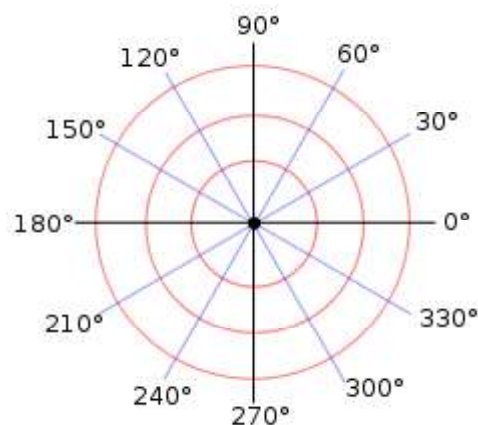
Fig. 4.5 *A Polar Grid Showing Different Angles Labelled in Degrees*

Example 1. Convert the point $(2, \pi/3)$ from polar coordinate to Cartesian coordinate.

Solution: Follow the steps given below for converting the points $(2, \pi/3)$ from polar coordinate to Cartesian coordinate.

Because, $\quad r = 2 \quad$ and $\quad \theta = \pi/3$

Now, $\qquad x = r \cos \theta$

$\qquad\qquad = 2 \cos \pi/3 = 2 \times 1/2 = 1$

And, $\quad y = r \sin \theta$

$$= 2 \sin \pi/3 = 2 \times \sqrt{3}/2 = \sqrt{3}$$

Hence, points $(2, \pi/3)$ in Cartesian coordinate is $(1, \sqrt{3})$.

Example 2. Convert the points $(-1, -1)$ into polar coordinates.

Solution: Follow the steps given below for converting the points $(-1, -1)$ into polar coordinates.

We will first obtain 'r' as follows.

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Now we will obtain 'θ' as follows.

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \pi/4$$

Though, this angle is not correct, since this value of angle 'θ' is for the first quadrant but the points that are given actually position in the third quadrant. To obtain the correct angle, we add 'π' to the equation as follows.

$$\theta = \pi/4 + \pi = 5\pi/4$$

Therefore, points $(-1, -1)$ into polar coordinates is written as $(\sqrt{2}, 5\pi/4)$.

The angle 'θ' can also be evaluated by taking 'r' as negative. When 'r' is negative then the points in the polar coordinates can be written as $(-\sqrt{2}, \pi/4)$.

Simple Polar Coordinate Graphs

We can draw the polar coordinate graphs using the following notations and equations.

Graph Lines: The following are the simple form of equations for polar graph.

1. For $\theta = \beta$

This line is formed by converting the equation to Cartesian coordinate form as,

$$\theta = \beta$$
$$\tan^{-1}(y/x) = \beta$$
$$(y/x) = \tan \beta$$
$$y = (\tan \beta)\, x$$

This line passes through the origin making angle 'β' on the positive x-axis.

Alternatively, this line passes through the origin having a slope as $\tan \beta$.

2. For $r \cos \theta = a$

This line is formed by simply converting to the form Cartesian coordinates as,

$$x = a$$

Because, $x = r \cos \theta$. The line formed is a vertical line.

3. For $r \sin \theta = b$

Similarly, this line is formed by converting to the form Cartesian coordinates as,

$$y = b$$

Because, $y = r \sin \theta$. The line formed is a horizontal line.

Circles: The following are the equations for circles with regard to the polar coordinates.

1. For $r = a$

The equation '$r = a$' specifies that the distance from the origin is 'a' irrespective of the type of angle, i.e., it can be defined as the circle which is centered at the origin having radius '$|a|$'.

2. For $r = 2a \cos \theta$

The equation '$r = 2a \cos \theta$' specifes that this is a circle having radius '$|a|$' and a center $(a, 0)$. In this equation if 'a' be negative then we will mark absolute values on the bar but not on the centre.

3. For $r = 2b \sin \theta$

The equation '$r = 2b \sin \theta$' specifes that this is a circle having radius '$|b|$' and a center $(0, b)$.

4. For $r = 2a \cos \theta + 2b \sin \theta$

The equations of above mentioned two conditions 2 and 3 are combined to complete the square and to obtain the circle having radius $\sqrt{a^2 + b^2}$ and the center at (a, b). This equation states the general equation of a circle which is not at all centred at the origin.

Cardioids and Limacons: The following are the three valid equations about Cardioids and Limacons.

1. Cardioids

$r = a \pm a \cos \theta$ and $r = a \pm a \sin \theta$

The graphs always have the origin and are ambiguously of heart shaped.

2. Limacons with an Inner Loop

$r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$ using $a < b$

These graphs also always have the origin and contain an inner loop.

3. Limacons without an Inner Loop

$r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$ using $a > b$

The graphs neither have the origin nor they have an inner loop.

Check Your Progress

1. What is polar axis?

2. Define the polar coordinates (r, θ) in a plane.

3. What does $r = 2b \sin \theta$ specify?

4.3 RADIUS OF CURVATURE IN POLAR COORDINATES

The term **'Curvature'** is specifically used for measuring how fast the direction is changed whenever we move a small distance along a curve. The direction can be assigned a numerical value termed as the **'angle of the tangent line'**. The curvature is measured on the basis of rate of change of this angle with reference to arc length.

Definition. Augustin-Louis Cauchy has defined the center of curvature **'C'** as the intersection point of two infinitely close normals to the curve, the **'radius of curvature'** as the distance from the point to C, and the curvature itself as the inverse of the radius of curvature. The curvature of a circle is therefore defined to be the reciprocal of the radius as,

$$\kappa = 1 / R$$

Where 'κ' (Kappa) represents curvature while 'R' represents the radius of the circle.

Definition 1. The reciprocal of the curvature of a curve is called the **'radius of curvature'** of the curve.

Definition 2. The **'radius of curvature'** of a curve at a point is the reciprocal of the curvature, i.e.,

Radius of Curvature or 'ρ' = 1 / Curvature = 1 / κ

Where 'ρ' represents the **radius of curvature**.

Definition. In the **polar coordinates '$r = r(\theta)$'**, the **'radius of curvature'** is represented as,

$$R = \frac{\left(r^2 + r_\theta^2\right)^{3/2}}{\left|r^2 + 2\,r_\theta^2 - r\,r_{\theta\theta}\right|},$$

Where $r_\theta = dr / d\theta$ and $r_{\theta\theta} = d^2r / d\theta^2$

Definition. If a **curve** is given by the **polar equation '$r = r(\theta)$'**, then the **curvature** is calculated by the formula,

$$\kappa = \frac{\left|r^2 + 2(r')^2 - rr''\right|}{\left[r^2 + (r')^2\right]^{\frac{3}{2}}}$$

The **radius of curvature** of a **curve** at any point is termed as the inverse of the curvature 'κ' of the curve and is represented as,

$$\rho = 1 / \kappa$$

Where 'ρ' represents the radius of curvature.

Example 3. For the parabola $y = x^2$ at the origin, find the curvature and radius of curvature.

Solution: We will first state the derivatives of the quadratic function as follows,

$$y' = (x^2)' = 2x$$
$$y'' = (2x)' = 2$$

Subsequently, the curvature of the parabola is obtained using the formula,

$$\kappa = \frac{y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}} = \frac{2}{\left[1 + (2x)^2\right]^{\frac{3}{2}}} = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}$$

Therefore, at the origin, i.e., at '$x = 0$':

Curvature 'κ' is,

$$\kappa\,(x = 0) = \frac{2}{(1 + 4 \cdot 0^2)^{\frac{3}{2}}} = 2$$

And the Radius of Curvature 'ρ' is,

$$\rho = 1 / \kappa = 1 / 2$$

Check Your Progress

4. Define curvature of a circle.

5. What is radius of curvature of a curve?

4.4 ASYMPTOTES

Definition. An **asymptote** of a **curve** is defined as a line which is **tangent** to the **curve** at a **point at infinity**.

Definition. An **asymptote** of a **curve '*y* = *f*(*x*)'** that has an infinite property is defined as a line such that the **distance** between the **points (*x*, *f*(*x*))** lying on the **curve** and the **line** approaches **zero** as the point moves along the **point at infinity**.

Definition. An **asymptote** is defined as a **line** or **curve** that approaches a given **curve arbitrarily closely**. This is shown in Figure 4.6.
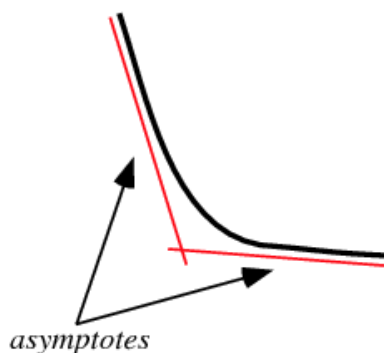
asymptotes

Fig. 4.6 *Asymptotes*

Basically, the asymptotes are categorized into three types as **horizontal** asymptotes, **vertical** asymptotes and **oblique** asymptotes. These asymptotes can be defined for the curves of the graph of a function *y* = *f*(*x*) as follows.

Vertical Asymptotes

These are the vertical lines which define that near it the function can grow without any bound. We can define the vertical asymptote of the form as the straight line for '*x* = *a*' of the graph for the function '*y* = *f*(*x*)' when in any case one of the below given conditions is true:

1. $\lim_{x \to a-0} f(x) = \pm\infty.$

2. $\lim_{x \to a+0} f(x) = \pm\infty.$

Alternatively, at the point '*x* = *a*' at least any one of the above defined limits essentially be equal to infinity.

The vertical asymptote is possible in the function in which at the points the graph of the function '*y* = 1/*x*' has '*x* = 0' as the vertical asymptote, as shown in the Figure 4.7.
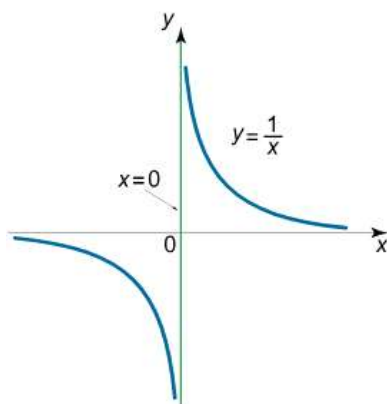
Fig. 4.7 *Vertical Asymptote*

As per the Figure 4.7, we can state that both the following limits, left and right, tend to infinity. Thus,

$$\lim_{x \to 0-0} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0+0} \frac{1}{x} = +\infty$$

Remember that there is no vertical asymptotes of a function that is continuous on the whole set of real numbers.

Oblique Asymptotes

These asymptotes have a non-zero but finite slope, i.e., when the graph of the function approaches it when 'x' tends to either '$+\infty$' or $-\infty$'.

In Figure 4.8, the straight line '$y = kx + b$' is termed as the oblique or slant asymptote for the graph of the specific function '$y = f(x)$' since '$x \to +\infty$' and when,

$$\lim_{x \to +\infty} [f(x) - (kx + b)] = 0.$$

In the same way, we can have the oblique asymptote as '$x \to -\infty$'.
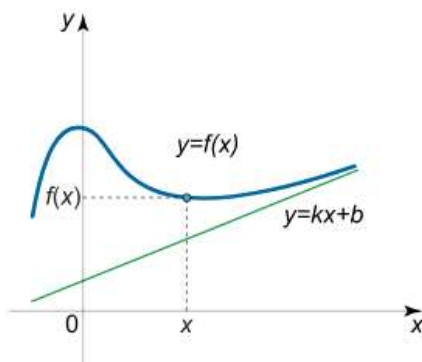
Fig. 4.8 *Oblique Asymptote*

The oblique asymptote can be different for the graph of the function '$y = f(x)$' for the conditions '$x \to +\infty$' and '$x \to -\infty$'.

Hence, in order to find out the oblique asymptotes we must separately consider both the above mentioned conditions.

The following theorem defines the 'k' and 'b', the coefficients of oblique asymptotes.

Theorem. A **straight line '$y = kx + b$'** is an **asymptote** of the **function '$y = f(x)$'** as '$x \to +\infty$' if and only if the following two limits are finite:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k \quad \text{and} \quad \lim_{x \to +\infty} [f(x) - kx] = b.$$

Proof.

Consider that the straight line '$y = kx + b$' is an asymptote for the graph of the function '$y = f(x)$' when '$x \to +\infty$', then the following given condition holds true:

$$\lim_{x \to +\infty} [f(x) - (kx + b)] = 0$$

Equivalently, we can state that,

$$f(x) = kx + b + \alpha(x)$$

Where, $\qquad \lim_{x \to +\infty} \alpha(x) = 0.$

Now, when both sides of the equation is divided by 'x' then we have,

$$\frac{f(x)}{x} = \frac{kx + b + \alpha(x)}{x}$$

$$\Rightarrow \qquad \frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha(x)}{x}$$

Accordingly, for the limit as '$x \to +\infty$', we get the following form of equations:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \left[k + \frac{b}{x} + \frac{\alpha(x)}{x} \right] = k,$$

$$\lim_{x \to +\infty} [f(x) - kx] = \lim_{x \to +\infty} [b + \alpha(x)] = b.$$

Assume that the finite limits exist, then

1. $\quad \lim_{x \to +\infty} \frac{f(x)}{x} = k$

2. $\quad \lim_{x \to +\infty} [f(x) - kx] = b.$

The limit defined in the above mentioned case 2 can be represented in the form,

$$\lim_{x \to +\infty} [f(x) - (kx + b)] = 0,$$

This satisfies the condition defined in the definition of the oblique asymptote. Therefore, the straight line '$y = kx + b$' is an asymptote of the function '$y = f(x)$'.

Hence, proved.

Horizontal Asymptotes

These are the horizontal lines which state that the graph of the function approaches when 'x' tends to either '$+\infty$' or $-\infty$'.

We can acquire a horizontal asymptote by defining the equation of the line '$y = b$', specifically when '$k = 0$'. The following theorem states the essential conditions required for the horizontal asymptote to exist.

Theorem. A straight line '$y = b$' is an asymptote of a function '$y = f(x)$' specifically as '$x \to +\infty$', if and only if the following limit is finite:

$$\lim_{x \to +\infty} f(x) = b.$$

Similar conditions apply for '$x \to -\infty$'.

Note: An asymptote is defined as a line which is precisely the graph of a function which approaches but under no circumstances touches.

4.4.1 Method of Finding Asymptotes

Any given rational function possibly will or will not have a vertical asymptote, based on the condition that the denominator is equals to '0', but it will definitely have a horizontal asymptote or an oblique asymptote.

The method used for finding the horizontal asymptote depends on by what method the degrees of the polynomials are compared in the numerator and denominator.

1. When both polynomials have the same degree then we divide the coefficients terms of highest degree.

2. When the polynomial in the numerator is comparatively of lower degree as that of the denominator, then the x-axis ($y = 0$) has the horizontal asymptote.

3. When the polynomial in the numerator is comparatively of higher degree as that of the denominator, then there will not be any horizontal asymptote.

For example, if

$$f(x) = 6x^2 - 3x + 4 / 2x^2 - 8$$

Then in this case both the polynomials are of 'Degree 2' hence the horizontal asymptote is at,

Either, $y = 6/2$

Or, $y = 3$

Example 4. Find the asymptotes for the function of a graph,

$$y = x / x + 1$$

Solution: For finding the the asymptotes follow the steps given below.

If '$x = -1$' then the function is assumed to be discontinuous. Certainly then,

1. $$\lim_{x \to -1-0} f(x) = \lim_{x \to -1-0} \frac{x}{x+1}$$

$$= \frac{-1}{(-1-0)+1}$$

$$= -1/-0 \quad = +\infty$$

2. $$\lim_{x \to -1+0} f(x) = \lim_{x \to -1+0} \frac{x}{x+1}$$

$$= \frac{-1}{(-1+0)+1}$$

$$= -1/+0 \quad = -\infty$$

Therefore, the equation '$x = -1$' specifies that it is the equation for vertical asymptote and the vertical asymptote exists.

For finding the horizontal asymptote we compute as follows:

$$\lim_{x \to \pm\infty} \frac{x}{x+1} = \lim_{x \to \pm\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Consequently, the curve has the horizontal asymptote and the equation for the horizontal asymptote graph is '$y = 1$', i.e., the horizontal asymptote exists.

The oblique asymptote does not exist at all. Let us prove this by computing the coefficients 'k' and 'b' as follows:

1. $$k = \lim_{x \to \pm\infty} \frac{y(x)}{x} = \lim_{x \to \pm\infty} \frac{x}{(x+1)x}$$

$$= \lim_{x \to \pm\infty} \frac{x}{x^2+x}$$

$$= \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1+\frac{1}{x}} = 0$$

2. $b = \lim\limits_{x \to \pm\infty} [y(x) - kx]$

$= \lim\limits_{x \to \pm\infty} \left(\dfrac{x}{x+1} - 0 \right)$

$= \lim\limits_{x \to \pm\infty} \dfrac{1}{1 + \frac{1}{x}} = 1.$

Thus, after computing the limits, we have obtained the equation as '$y = 1$', i.e., the horizontal asymptote. The oblique asymptote does not exist.

Therefore, for the graph of the function we obtained,

Vertical Asymptote = '$x = -1$'

Horizontal Asymptote = '$y = 1$'

Oblique Asymptote = Does Not Exist

Following graph in Figure 4.9 is of the horizontal and vertical asymptotes.
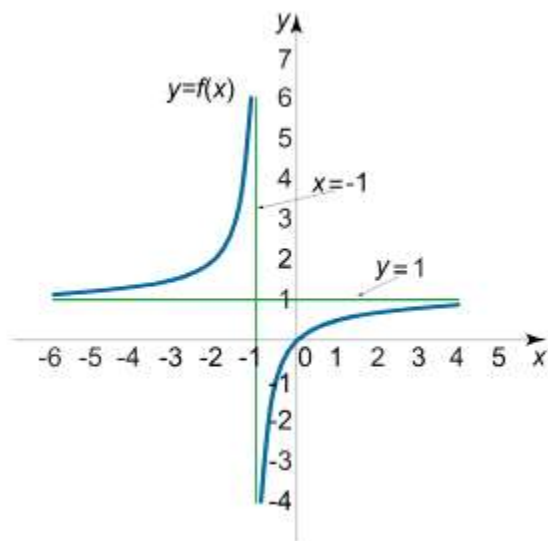
Fig. 4.9 *Horizontal and Vertical Asymptotes*

Check Your Progress

6. What is an asymptote?

7. What are oblique asymptotes?

4.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. The ray that originates from the pole towards the reference direction is termed as the polar axis.

2. In the plane we can define the polar coordinates (r, θ) through,

   $r =$ Distance from the Origin or Radial Coordinate

   $\theta =$ Polar Angle or Angular Coordinate

   $\theta \in [0, 2\pi) =$ Counterclockwise Angle

3. The equation '$r = 2b \sin \theta$' specifes that this is a circle having radius '$|b|$' and a center $(0, b)$.

4. The curvature of a circle is therefore defined to be the reciprocal of the radius.

5. The reciprocal of the curvature of a curve is called the radius of curvature of the curve.

6. An asymptote of a curve is defined as a line which is tangent to the curve at a point at infinity.

7. These asymptotes have a non-zero but finite slope, i.e., when the graph of the function $y = f(x)$ approaches it when 'x' tends to either '$+\infty$' or $-\infty$'.

4.6 SUMMARY

- The polar coordinate are determined through a distance with regard to 'a reference point and an angle from a reference direction'. The reference point is termed as the 'pole'. The ray that originates from the pole towards the reference direction is termed as the 'polar axis'. The distance from the pole is termed as the 'radius' and the 'angle' is termed as the 'angular coordinate', 'polar angle'.

- $x = r \cos\theta$, $y = r \sin\theta$, $r^2 = x^2 + y^2$ and $\tan \theta = y/x$ where (r, θ) are polar coordinates and (x, y) are Cartesian coordinates.

- The curvature of a circle is therefore defined to be the reciprocal of the radius as, $\kappa = 1 / R$ where 'κ' (Kappa) represents curvature while 'R' represents the radius of the circle.

- The reciprocal of the curvature of a curve is called the 'radius of curvature' of the curve.

- The 'radius of curvature' of a curve at a point is the reciprocal of the curvature.

- In the polar coordinates '$r = r(\theta)$', the 'radius of curvature' is represented as,

$$R = \frac{(r^2 + r_\theta^2)^{3/2}}{\left|r^2 + 2\,r_\theta^2 - r\,r_{\theta\theta}\right|}, \text{ Where } r_\theta = dr\,/\,d\theta \text{ and } r_{\theta\theta} = d^2 r\,/\,d\theta^2$$

- If a curve is given by the polar equation '$r = r(\theta)$', then the curvature is calculated by the formula,

$$\kappa = \frac{\left|r^2 + 2(r')^2 - rr''\right|}{\left[r^2 + (r')^2\right]^{\frac{3}{2}}}$$

- The radius of curvature of a curve at any point is termed as the inverse of the curvature 'κ' of the curve and is represented as, $\rho = 1\,/\,\kappa$ where 'ρ' represents the radius of curvature.

- An asymptote of a curve is defined as a line which is tangent to the curve at a point at infinity. The asymptotes are categorized into three types as horizontal asymptotes, vertical asymptotes and oblique asymptotes.

4.7 KEYWORDS

- **Polar coordinates:** the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

- **Cartesian coordinates:** numbers which indicate the location of a point relative to a fixed reference point (the origin), being its shortest (perpendicular) distances from two fixed axes (or three planes defined by three fixed axes) which intersect at right angles at the origin.

- **Polar angle:** In the plane, the polar angle theta is the counterclockwise angle from the x-axis at which a point in the xy-plane lies.

- **Radius:** a straight line from the centre to the circumference of a circle or sphere.

- **Asymptote:** an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity.

4.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Convert the point $(5, -2)$ into polar coordinates.
2. Convert the point $(3, 2\pi/3)$ from polar coordinate to Cartesian coordinate.

3. Graph the polar equation $r = 1 + \sin \theta$.

4. Graph the polar equation $r = 1 + \cos \theta$.

Long Answer Questions

1. Discuss polar coordinates with help of a graph.

2. How are the polar coordinates converted to Cartesian coordinates?

3. Expalin the method used for finding the horizontal asymptote.

4. Find the asymptotes for the function of a graph,

   $y = 2x / 2x + 1$

5. Briefly describe vertical asymptote.

6. Briefly describe horizontal asymptote.

4.9 FURTHER READINGS

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BLOCK - II

ENVELOPES, EVOLUTES AND INTEGRALS

UNIT 5   TANGENTS, CURVATURE, ENVELOPES AND EVOLUTES

Structure

5.0   INTRODUCTION

This unit describes how differentiation can be used to calculate the equations of the tangent and normal to a curve. The tangent is a straight line which just touches the curve at a given point. The normal is a straight line which is perpendicular to the tangent. Aspects of curvature are also discussed in this unit. Curvature is a measure of how much the curve deviates from a straight line. In other words, the curvature of a curve at a point is a measure of how much the change in a curve at a point is changing, meaning the curvature is the magnitude of the second derivative of the curve at given point. In the end, you will learn about envelopes and evolutes. The idea of an envelope plays an important role in determining solution to fully nonlinear scalar partial differential equation.

5.1   OBJECTIVES

After going through this unit, you will be able to:

- Define tangents, normal to a curve and normal angle of intersection
- Calculate the equation of the tangent to a curve at a given point

- Calculate the equation of the normal to a curve at a given point
- Know about curvature and its angles
- Find envelopes and evolutes for a curve

5.2   TANGENTS AND NORMAL ANGLE OF INTERSECTION

Definition. The **tangent line** or simply tangent to a **plane curve** at **any given point** is the **straight line** that simply touches the **curve at that point**.

Definition. At a **given point on a curve**, the **gradient of the curve** is **equal to** the **gradient of the tangent** to the **curve**. This is shown in Figure 5.1.
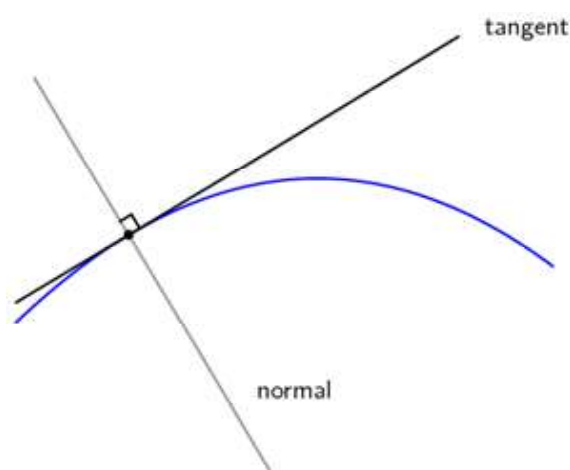
Fig. 5.1 *Tangents and Normal Angle of Intersection*

Fundamentally, the derivative or gradient function defines the gradient of a curve at some particular point on the stated curve. Additionally, it also defines the gradient of a tangent to a curve at some particular point on the stated curve.

Definition. The **normal to a curve** is the **line perpendicular** to the **tangent** to the **curve** at a **given point**. Thus,

$$m_{\text{tangent}} \ \times \ m_{\text{normal}} \ = \ \text{`}-1\text{'}$$

Equation of a Normal to a Curve

The term **'normal'** specifies that it is either **'perpendicular'** or **'at right angles'**. In Figure 5.2, the **normal** is represented by a **line** which is at **right angles** or **perpendicular** to the **'tangent'**.
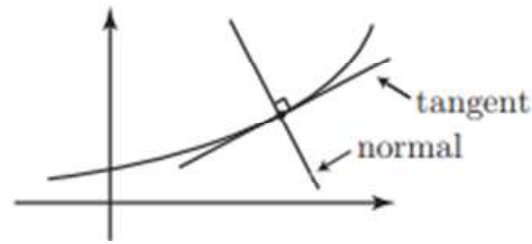
***Fig. 5.2** Normal Line at Right Angles to the Tangent*

To find the equation for the normal with regard to a curve, consider two lines which are at right angles (perpendicular) to each other and have the gradients as m_1 and m_2, respectively. The product of the gradients m_1 and m_2 has to be '– 1', i.e.,

$$m_1 \times m_2 = \text{'– 1'}$$

Definition of Normal to a Curve

The normal to a curve at some particular point '*P*' can be defined as the straight line that passes through that point and is at right angles or perpendicular to the tangent of the curve at that specified point (Refer Figure 5.3).

Definition of Tangent to a Curve

To define **a tangent** to **a curve**, consider that **'*P*'** is some particular point on a curve and let **'*Q*'** be some other point that lies in the region near point '*P*', as shown in Figure 5.3. Moreover, the point '*Q*' may possibly be considered on either side of point '*P*'. When '*Q*' tends towards '*P*', then the straight line '*PQ*' normally tends in the direction of a definite straight line '*TP*' which passes through '*Q*'. This **straight line** is termed as the **'tangent'** to the **curve** at the specified point **'*P*'**.
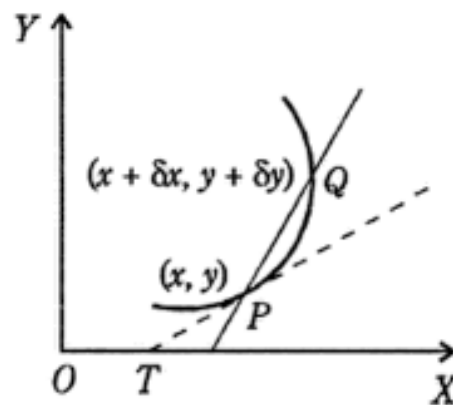
***Fig. 5.3** Tangent to a Curve*

Equation of the Tangent

To define the equation of the curve, consider that the function '$y = f(x)$' be the Cartesian equation of a specific curve. Let a point 'P' is specified on the coordinates (x, y) on the stated curve. Now, on this curve in the region near point 'P' consider a point $Q(x + \delta x, y + \delta y)$, as shown in Figure 5.3. When (X, Y) are defined as the current coordinates of a specified point on the 'chord PQ', then in such a case we have the following equation for the chord 'PQ'.

$$Y - y = \frac{(y + \delta y) - y}{(x + \delta x) - x} (X - x)$$

Or,
$$Y - y = \frac{\delta y}{\delta x} (X - x).$$

Consequently, when 'Q' tends towards 'P', then $\delta x \to 0$ and chord PQ will tend towards the tangent at 'P'. Hence, in such a case the above equation will have the form,

$$Y - y = \frac{dy}{dx} (X - x).$$

Because,
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore, the equation of the tangent to the curve $y = f(x)$ at the point (x, y) is as follows,

$$Y - y = \frac{dy}{dx} (X - x).$$

Case 1: Similarly, for finding the equation of the tangent for the curve $y = f(x)$ at the specified points (x_1, y_1), first we have to find the value of dy/dx for the curve at the specified point (x_1, y_1). The equation of the tangent at the specified point (x_1, y_1) is as follows,

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1),$$

Here, (x, y) are defined as the current coordinates of some particular point on the tangent.

Case 2: The equation of the tangent can also be defined when the given equations of the curve are in the form parametric Cartesian as follows,

$$x = f(t) \qquad \text{and} \qquad y = \phi(t)$$

Then accordingly, we have the equation,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}$$

Therefore, on the given curve, the equation of the tangent at some particular point '*t*' can be defined as follows,

$$Y - \phi(t) = \frac{\phi'(t)}{f'(t)}[X - f(t)].$$

Description of *dy/dx* in Accordance with Geometry

To geometrically define *dy/dx*, consider '*P*' as some specific point (x, y) for the given curve $y = f(x)$.

Assume that '*x*' increases in the positive direction because of the tangent at '*P*'. Let this positive direction caused by the tangent at '*P*' makes an angle 'ψ' along with the specified positive direction on the *x*-axis, as shown in the Figure 5.4. Then the equation for the tangent at '*P*' can be given as follows,

$$Y - y = \frac{dy}{dx}(X - x)$$

Then, $Y = \left(\frac{dy}{dx}\right)X + \left(y - x\frac{dy}{dx}\right)$ ...(1)

We can define that the Equation (1) is of the form,

$$Y = mX + c \qquad \qquad \qquad \text{...(2)}$$

As per the Equation (2) specifies that it is for the straight line with gradient as '*m*', specifically with the positive direction on *x*-axis, the line forms an angle with tangent as '*m*'.

Consequently, on equating Equations (1) and (2), we have the following resultant equation:

dy/dx = tan ψ
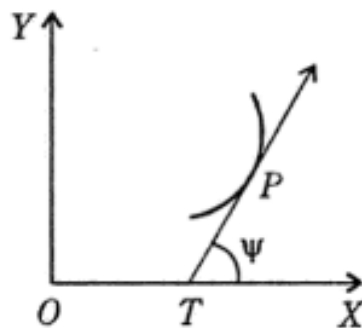
Fig. 5.4 *Tangent at 'P' makes an Angle 'ψ'*

Definition. The differential coefficient *dy/dx* at some particular point (x, y) on the curve $y = f(x)$ is equal to the tangent of the angle which the positive direction of the tangent at '*P*' to the curve makes with the positive direction of the *x*-axis.

Tangents Parallel to Coordinate Axes

Definition. While the tangent at some specific point is parallel to the axis of '*x*', then it can be defined that '$\psi = 0$' specifically '$\tan \psi = 0$' and hence at that specific point '$dy/dx = 0$'. Alternatively, when the tangent is parallel to the axis of '*y*' or it is perpendicular to the axis of '*x*' then we can state that at the stated point,

$$\psi = \pi/2$$

Specifically, $\qquad \tan \psi = \tan (\pi/2) = \infty$

Therefore, $\qquad dy/dx = \infty \quad$ or $\quad dx/dy = 0$

Equation of Normal

Consider that on the curve $y = f(x)$, when '*P*' is some specific point (x, y) then the equation of the tangent at '*P*' of the form,

$$Y - y = \frac{dy}{dx} (X - x)$$

Or else, $\qquad Y = \left(\frac{dy}{dx}\right) X + \left(y - x \frac{dy}{dx}\right)$

Consequently, the gradient can be defined as 'dy/dx' of the tangent at '*P*'.

Additionally, when the gradient of normal at the point '*P*' is defined by '*m*' then,

$$m \cdot dy/dx = -1$$

Or, $\qquad m = -1 / (dy/dx) = -(dx/dy)$

Therefore, the equation of normal for the specified curve at the point '*P*' is of the form,

$$Y - y = -\frac{dx}{dy} (X - x)$$

Or, $\qquad \frac{dy}{dx} (Y - y) + (X - x) = 0$

Note: When the equation of a specified curve has the form $f(x, y) = 0$, then we have the equation,

$$dy/dx = -((\partial f/\partial x) / (\partial f/\partial y))$$

Angle of Intersection

Definition. The **angle of intersection** of **two curves** is specified as the **angle** between the **tangents** of the **two curves** at the **point of intersection**. For determining or defining the angles of intersection for the two specified curves, we can state,

$$f(x, y) = 0 \qquad \qquad \qquad \qquad \text{…(3)}$$

And, $\qquad \phi(x, y) = 0 \qquad \qquad \qquad \qquad \text{…(4)}$

By simultaneously solving Equations (3) and (4), we obtain the points of intersection for the Equations (3) and (4).

If one of the points of intersection is taken as (x_1, y_1), then for finding the angle of intersection at the point (x_1, y_1) we will first obtain the slopes of the two curves as m_1 and m_2 for the tangents of both the curves at the points (x_1, y_1) as given below.

Since, $m_1 = (dy/dx)$ at the points (x_1, y_1) for the curve of Equation (3)

And, $m_2 = (dy/dx)$ at the points (x_1, y_1) for the curve of Equation (4)

Subsequently, we can define that:

When $m_1 = m_2$, then the angle of intersection will be 0°.

Alternatively, we can state that '$m_1 = m_2$' is the necessary condition when the two curves will touch each other at the point of intersection.

When $m_1 = \infty$, and $m_2 = 0$, then the angle of intersection will be 90°.

When $m_1 \, m_2 = -1$, then the angle of intersection for a second time will be 90° and the intersection of the two curves will be orthogonal or at the right angles.

In the all remaining conditions, we can state that the acute angle that exists between the tangents will be equivalent to,

$$\tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

Let us understand the concept of the angle of intersection of two curves with the help of an example. Consider that the given two curves form the angle of intersection as 'θ' at the point of intersection 'P', as shown in Figure 5.5, then we can state that,

$$\theta = \left| \psi_1 - \psi_2 \right|$$

Or, $\theta = \tan^{-1} \left| (m_1 - m_2) / (1 + m_1 m_2) \right|$

The angles 'ψ_1' and 'ψ_2' are the angles specifically made by the tangents towards the two curves on the point of intersection along with the positive direction of the x-axis. Subsequently, m_1 and m_2 can be defined as gradients of the two specified curves on the point of intersection, viz.,

$$m_1 = \tan \psi_1 \qquad \text{and} \qquad m_2 = \tan \psi_2$$
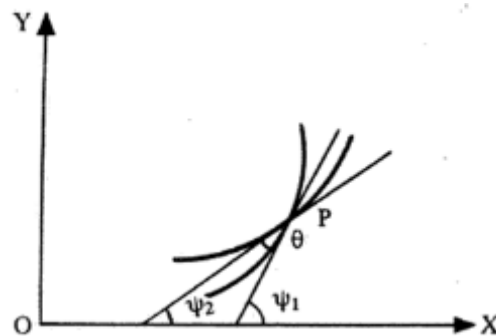
***Fig. 5.5** Angle of Intersection of Two Curves*

Example 1. Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Solution: To find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$, we will first solve these given equation as follows,

$$y^2 = x$$

And, $\qquad x^2 = y$

$\Rightarrow \qquad x^4 = x \qquad$ or $\qquad x^4 - x = 0$

$\Rightarrow \qquad x(x^3 - 1) = 0$

$\Rightarrow \qquad x = 0, \ x = 1$

Consequently, $\qquad y = 0, \ y = 1$

Hence, the points of intersection are,

$$(0, 0) \quad \text{and} \quad (1, 1)$$

Additionally,

$$y^2 = x$$

$\Rightarrow \quad y(dy/dx) = 1$

$\Rightarrow \quad dy/dx = 1/2y$

Similarly,

$$x^2 = y$$

$\Rightarrow \quad dy/dx = 2x$

Therefore,

At $(0, 0)$, for the curve '$y^2 = x$', the slope of the tangent is parallel to y-axis and for the curve '$x^2 = y$', the slope of the tangent is parallel to x-axis.

$\Rightarrow \quad$ Angle of intersection $= \pi/2$

At $(1, 1)$, for the curve '$y^2 = x$', the slope of the tangent is equal to $1/2$ and for the curve '$x^2 = y$', the slope of the tangent is equal to 2.

Hence,

$$\tan \theta = \left| \frac{2 - \frac{1}{2}}{1 + 1} \right| = \frac{3}{4}.$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 2. Analyze the condition when the curves $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $xy = c^2$ will orthogonally intersect each other.

Solution: To evaluate the condition that the given curves intersect orthogonally, assume that the intersection of curves takes place at (x_1, y_1).

Hence, on equating the given equation we have,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$\Rightarrow$ The slope of the tangent of the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point of intersection, say (m_1) will be,

$$(m_1) = \frac{b^2 x_1}{a^2 y_1}$$

Further, for the curve $xy = c^2$ on equating we have,

$$xy = c^2$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad dy / dx = (-y) / x$$

$\Rightarrow$ The slope of the tangent of the curve $xy = c^2$ at the point of intersection, say (m_2) will be,

$$m_2 = (-y_1) / x_1$$

Applying the formula for the curves to intersect orthogonally,

$$m_1 \, m_2 = -1$$

$$\Rightarrow \quad b_2 / a_2 = 1 \quad \text{or} \quad a_2 - b_2 = 0$$

Check Your Progress

1. What is a tangent line?
2. What is normal to a curve?
3. What is angle of intersection?

5.3 CURVATURE

The term **'Curvature'** is defined as the bending of curves at the different varied points. Assume that a curve has a point '*P*' on it, as shown in Figure 5.5. Now, let '*Q*' be another point near the point '*P*' on the curve, i.e., both are adjacent points.

Consider a fixed point '*A*' on the same plane curve on which we have assumed points *P* and *Q*. Let at point '*R*' the tangents at *P* and *Q* meet such that angle '$\delta\psi$' is at *R* when due to tangents at *P* and *Q* the angles 'ψ' and '$\psi + \delta\psi$' are made on the *x*-axis.

Now, assume that,

Arc *AP* = S

And   Arc *AQ* = S + δS

Therefore,   Arc *PQ* = δS

Because,

'ψ' and '$\psi + \delta\psi$' are the angles made due to tangents at *P* and *Q* on the *x*-axis

Consequently, '$\delta\psi$' is termed as the **total curvature** of the Arc *PQ*.

And,   '$\delta\psi/\delta$S' is termed as the **average curvature** of the Arc *PQ*.

Hence, the curvature of the curve at point *P* is defined as,

$$\lim_{\delta S \to 0} \frac{\delta\psi}{\delta S} = \frac{d\psi}{dS} :$$
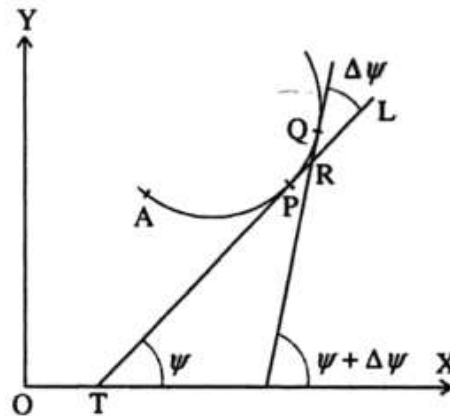
Fig. 5.5 *Curvature and its Angles*

5.4   ENVELOPES AND EVOLUTES

An **'Envelope'** can be defined as a **'curve'** that touches every single member of the family of curves or lines. The axes of the circles are termed as the envelopes for the methods of circles represented as,

$(x - a)^2 + (x - a)^2 = a^2$

An **'Evolute'** are the envelope of the normals to the specified curve. It can also be assumed as the **'locus of the centres of curvature'**.

Definition 'Evolute'. According to the differential geometry of curves, we can define that the **evolute** of a curve is the locus of all its centers of curvature, i.e., when the center of curvature of each point on a curve is drawn, then the resulting subsequent shape will be the evolute of that curve. Hence, "*The evolute of a circle is a single point at its center. Equivalently, an evolute is the envelope of the normals to a curve*".

Let us first understand the concept of family of curves. A curve is represented by an equation which has the form,

$$f(x, y, \alpha) = 0 \qquad \qquad \dots(5)$$

In this equation 'α' is a constant. If we consider α as a parameter, specifically when α takes all of the real values then in this case Equation (5) will be the equation of **'one parameter family of curves with α parameter'**. We can provide different values to α in order to obtain various other members of the curve family, i.e., for different values of 'α' we obtain different curves.

Note: For any specific curve that belongs to this specific family the values of α will remain constant however it will modify from one curve to another curve. In the equation, every constant of the specified curve may not be a parameter.

Notation

1. The family of curves with one or single unique parameter 'α' is represented by, $f(x, y, \alpha) = 0$

2. The family of curves with two unique parameters 'α, β' is represented by,

$$f(x, y, \alpha, \beta) = 0$$

Consider that a family of circles (Refer Figure 5.6) has the following form of equation,

$$x^2 + y^2 - 2ax = 0$$

Figure 5.6 illustrates the family of circles in which all the circles have their centres on the *X*-axis and all the circles pass through the origin '*O*'.

In the equation '$x^2 + y^2 - 2ax = 0$', '*a*' represents a set of circles having centres on the *X*-axis and passing through the origin '*O*'.

In general, the equation of the plane curve comprises of two existing coordinates '*x, y*' of the specified point (x, y) on the certain curve and definite constants. These definite constants precisely define the 'size or extent, shape or nature and position of that curve'.
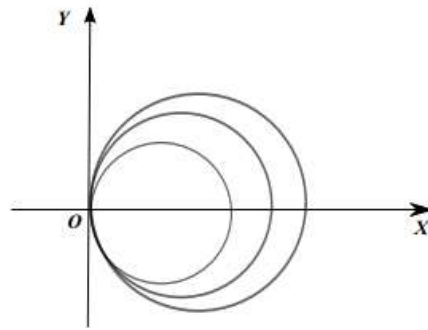
Fig. 5.6 *Family of Circles*

Definition 'Envelope'. An **'Envelope'** can be defined as a **'curve'** that touches every single member of the family of curves or lines. The axes of the circles are termed as the envelopes for the methods of circles represented as,

$$(x - a)^2 + (x - a)^2 = a^2$$

The essential and sufficient condition for an envelope to exist for a family of curves '$f(x, y, \alpha) = 0$', here 'α' is a parameter, is

$$\frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0$$

Evolute and Involute

Definition. The **involute** of a circle is defined as the **curve** for which all the **normals are tangent** to a **fixed circle**. Conversely, when an involute of a specified circle goes around the centre of the generating circle, then every tangent to the circle will always remain normal to the involute.

Let us understand the concept with the help of an example. Select the point on a curve and attach a string to it. Now, extend the string such that it remains tangent towards the curve on the point from where it is attached. Then keeping the string constantly stretched, gale the string up. Refer Figure 5.7 which illustrates this process for a circle. The locus of points that are drawn by the end of the stretched string is termed as the **'involute of the original definite curve'** while the original definite curve itself is termed as **'the evolute of its involute'**.
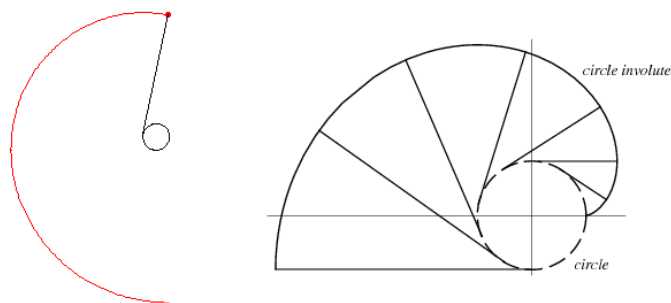
Fig. 5.7 *Involute of the Original Definite Circle*

Even though a curve represents a unique evolute, but it also considerably denotes numerous involutes that corresponds to the various options selected as of initial point. Therefore, "*An involute can be defined as some specific curve which is orthogonal to all the tangents to a specified assumed curve*".

5.4.1 Working Method To Find Envelope And Involutes

Consider that the family of curves has the equation of the form,

$$F(x, y, \alpha) = 0 \qquad \qquad \ldots(6)$$

Assume that '*P*' is the point of intersection of the two specified curves having the equation of the form,

$$F(x, y, \alpha) = 0 \quad \text{and} \quad F(x, y, \alpha + \delta\alpha) = 0 \qquad \ldots(7)$$

Here, 'α' and '$\alpha + \delta\alpha$' are the parameters of the curve family. In the curves of Equation (7) the coordinates of the points of intersection satisfy the equation of the form,

$$F(x, y, \alpha) = 0$$

And, $\quad F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha) = 0$

Consequently, we have the equations,

$$F(x, y, \alpha) = 0$$

And, $\quad (F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha)) / \delta\alpha = 0$

When we take limits as '$\delta\alpha \to 0$', then the coordinates for the position of points of intersection of the Equation (5.7) curves will satisfy the following equation:

$$F(x, y, \alpha) = 0$$

And, $\quad \delta F(x, y, \alpha) / \delta\alpha = 0 \qquad \qquad \ldots(8)$

Therefore, for all values of 'α', on the envelope the coordinate points satisfy the Equation (8). On eliminating 'α' in the Equation (8) we obtain the equation of the envelope for the family of curves.

Let us understand how these equations are generalized.

Assume that the family of curves has the equation of the form whose **envelope** is to be estimated,

$$F(x, y, \alpha) = 0 \qquad \qquad \ldots(9)$$

Now consider any two members of the family of curves whose equations are,

$$F(x, y, \alpha) = 0 \qquad \qquad \ldots(10)$$

And $\quad F(x, y, \alpha + \delta\alpha) = 0 \qquad \qquad \ldots(11)$

It is obvious that the points of intersection of the Equations (10) and (11) satisfies the equation of the form,

$$F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha) = 0$$

Or, $(F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha)) / \delta\alpha = 0$

Continuing to limits as '$\delta\alpha$' tends to '0', i.e., taking limit as '$\delta\alpha \to 0$', it is observed that the limit position 'P' of the specified point is common to the Equations (10) and (11) which satisfy the equation of the form,

$$\frac{\partial}{\partial\alpha} F(x, y, \alpha) = 0 \qquad \dots(12)$$

However, the coordinates of 'P' too satisfy the Equation (10), because P lies on it.

Consequently, on eliminating 'α' between Equations (9) and (12) will provide an equation of the form,

$$\phi(x, y) = 0 \qquad \dots(13)$$

The Equation (13) is termed as the **equation** of the **envelope**.

Additionally, the following form of equations are termed as the parametric equations for the envelope. The equation of envelope is obtained by eliminating 'α' between the equations.

$$F(x, y, \alpha) = 0$$

$$\frac{\partial}{\partial\alpha} F(x, y, \alpha) = 0$$

Common Guidelines to Find Envelope

Step 1. Differentiate the equation with regard to the '*variable parameter*' and consider all the remaining measures as constants that are involved in the specified equation.

Step 2. Elucidate the outcome or the result and also the given specified equation for the family of curves for 'x' and 'y' by defining the parameter. The resultant explanations signify the parametric equations of the envelope.

Definition of Variable Parameter: Any system of curves formed using this method is termed as the family of curves, and the quantity 'α', which is constant for any one curve but changes in passing from one curve to another curve, is termed as variable parameter.

Example 3. Show that the involute of the specified circle having radius R is centered at the origin, when the specified curve given by the following form of equations is of the evolute,

$$\xi = R \cos t, \ \eta = R \sin t$$

Solution: To prove that the involute of the circle having radius R is centered at the origin, we have the form of equations of the evolute as,

$$\xi = R \cos t, \ \eta = R \sin t$$

These equations are termed as the parametric equations of the specified circle having radius R and is centered at the origin.

Now using the Cartesian coordinates (ξ, η), we can write the equations for the evolute as follows,

$$\xi^2 + \eta^2 = R^2$$

The involute of the specified circle will be of spiral shape, as shown in Figure 5.8. Fundamentally, it defines the trajectory of some specific point of a straight line, that rolls or moves alongside the circumference in specified direction.
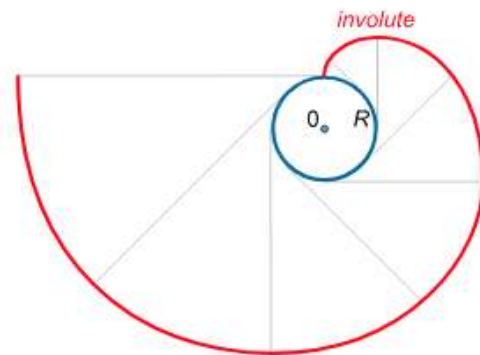
***Fig. 5.8** Involute*

Example 4. Find the envelope for the equation $y = mx + (a/m)$, where m is the parameter.

Solution: We have the equation of the form,

$$y = mx + (a/m) \qquad \qquad \dots(14)$$

Now, partially differentiate Equation (14) with respect to m to obtain the equation,

$$0 = x - (a/m^2) \implies m^2 = a/x \qquad \qquad \dots(15)$$

In order to obtain the required equation for the envelope, from Equations (14) and (15) we will eliminate the parameter m.

On squaring Equation (14) we will have,

$$y^2 = m^2x^2 + (a^2/m^2) + 2ax \qquad \qquad \dots(16)$$

Placing $m^2 = a/x$ in the above Equation (16) we obtain,

$$y^2 = x^2 (a/x) + a^2 (x/a) + 2ax$$

$$\implies \quad y^2 = 4ax \qquad \qquad \dots(17)$$

The resultant Equation (17) is the equation of the envelope which is a parabola.

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Check Your Progress

4. Define the term curvature.

5. What is an evolute?

6. What is an envelope?

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5.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The tangent line or simply tangent to a plane curve at any given point is the straight line that simply touches the curve at that point.

2. The normal to a curve at some particular point can be defined as the straight line that passes through that point and is at right angles or perpendicular to the tangent of the curve at that specified point

3. The angle of intersection of two curves is specified as the angle between the tangents of the two curves at the point of intersection.

4. The term curvature is defined as the bending of curves at the different varied points.

5. An evolute of a curve is the locus of all its centers of curvature.

6. An 'Envelope' can be defined as a 'Curve' that touches every single member of the family of curves or lines.

5.6 SUMMARY

- The tangent line or simply tangent to a plane curve at any given point is the straight line that simply touches the curve at that point.

- At a given point on a curve, the gradient of the curve is equal to the gradient of the tangent to the curve.

- The normal to a curve is the line perpendicular to the tangent to the curve at a given point.

- If two lines, with gradients m_1 and m_2 are at right angles then $m_1 m_2 = -1$

- The equation of the tangent to the curve $y = f(x)$ at the point (x, y) is as follows,

$$Y - y = \frac{dy}{dx}(X - x).$$

- The differential coefficient dy/dx at some particular point (x, y) on the curve $y = f(x)$ is equal to the tangent of the angle which the positive direction of the tangent at 'P' to the curve makes with the positive direction of the x-axis.

- While the tangent at some specific point is parallel to the axis of 'x', then it can be defined that '$\psi = 0$' specifically '$\tan \psi = 0$' and hence at that specific point '$dy/dx = 0$'.

- While the tangent at some specific point is parallel to the axis of 'x', then it can be defined that '$\psi = 0$' specifically '$\tan \psi = 0$' and hence at that specific point '$dy/dx = 0$'. Alternatively, when the tangent is parallel to the axis of 'y' or it is perpendicular to the axis of 'x' then we can state that at the stated point, $\psi = \pi/2$

- The equation of normal for the specified curve at the point 'P' is of the form,

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

- $\theta = \tan^{-1}\left|(m_1 - m_2)/(1 + m_1 m_2)\right|$, $m_1 = \tan \psi_1$ and $m_2 = \tan \psi_2$

- An 'Envelope' can be defined as a 'curve' that touches every single member of the family of curves or lines.

- An 'Evolute' are the envelope of the normals to the specified curve. It can also be assumed as the 'locus of the centres of curvature'.

5.7 KEY WORDS

- **Tangent:** The tangent line to a plane curve at a given point is the straight line that "just touches" the curve at that point.

- **Normal line:** The normal line is defined as the line that is perpendicular to the tangent line at the point of tangency.

- **Angle of intersection:** The angle of intersection of two curves is the angle between their tangent vectors at that point.

- **Gradient:** The gradient is a multi-variable generalization of the derivative. While a derivative can be defined on functions of a single variable, for functions of several variables.

5.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Find the equation of the tangent for the curve $y = f(x)$ at the specified points (x_1, y_1),

2. Find the equation of the tangent for a curve in the form of parametric Cartesian $x = f(t)$ and $y = \phi(t)$.

3. How is tangents parallel to coordinate axes?

4. Define the Evolute and Involute.

Long Answer Questions

1. Derive the equation of normal for the curve $y = f(x)$.

2. Find the angle of intersection of the curves $x = 3 + y^2$ and $y = 4 - x^2$.

3. Explain working method to find envelope and involutes.

4. Discuss the parametric equations for the envelope.

5. Find the equation of the normal lines to the curve $x^2 - y^2 = 4$ which are parallel to the line $x + 3y = 4$.

6. Find the envelope of the family of lines given by $= mx \pm a\sqrt{1 + m^2}$. $F(x, y; m) \equiv y - mx \pm \sqrt{1 + m^2} = 0$.

7. Briefly describe envelops.

5.9 FURTHER READINGS

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UNIT 6   INTEGRATION

Structure

6.0   INTRODUCTION

In this unit, you will learn about integration by the method of substitution. Using the fundamental theorem of calculus often requires finding an antiderivative. Therefore, integration by substitution is an important tool in mathematics. The idea of this method is to define a new variable which will allow the difficult starting integrand to be changed from the old variable to a new integrand which is in terms of the new variable. Integration by substitution is the counterpart to the chain rule for differentiation.

6.1   OBJECTIVES

After going through this unit, you will be able to:

- Evaluate an integral by making a substitution
- Identify appropriate substitutions to carry out integration

6.2   INTEGRATION – SUBSTITUTION METHODS

In calculus, the term **integration** is referred as the method of computing an **integral**, and the approximation or estimation of an integral is referred as the **numerical integration**. The **integral** of a **function** is approximated with regard to 'x', i.e., estimating an area from the curve to the x-axis. Integration process is specifically used for finding or approximating central points, areas, volumes, etc.

Usually, the **integral** is also sometimes termed as the **anti-derivative**, since **integration** is the **reverse process** of **differentiation**, i.e., **"Integration Reverses the Act of Differentiation"**. For example, differentiating x^3 provides $3x^2$ whereas integrating $3x^2$ provides x^3.

Characteristically,

Differentiation $\rightarrow$ Approximates the Slopes of Curves.

Integration $\rightarrow$ Approximates the Areas Under Curves.

Precisely, an integral can be defined as a mathematical object which is uniquely interpreted either as an area under the curve or as a generalization of area (Refer Figure 6.1). Therefore, the integrals and the derivatives are defined as the essential objects of calculus.
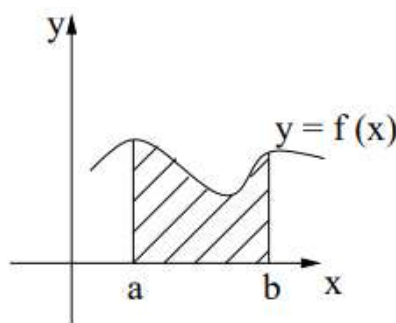
Fig. 6.1 Area Under Curve

Figure 6.1 illustrates the area under the curve that is to be approximated between,

$$(a, b) = \int_a^b f(x)\, dx$$

Terminology and Notation

$$\int_a^b f(x)\, dx$$

This notation is read as the **INTEGRAL** of '*f*' from '*x = a*' to '*b*'.

Following are some common notations that are used for computing integration.

$f(x)$	$\int f(x)\, dx$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\log \sec x$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$1/x$	$\log x$

Definite and Indefinite Integrals

The integrals are of two types, indefinite and definite.

Indefinite Integral: When it is not definite or specified how to start and end the process of integration then the **integral** is termed as **'indefinite integral'**.

For example, the following is the indefinite integral which is termed as the **'function of *x*'**.

$$\int f(x)\, dx.$$

Definite Integral: When it is definite or specific that how to start and end the process of integration, i.e., the **upper** and **lower limits** of **integration** (*b* and *a*) are defined then the **integral** is termed as **'definite integral'**.

For example, the following is the definite integral which is termed as the **'number'**.

$$\int_a^b f(x)\, dx$$

6.2.1 Integration by Substitution

In calculus, fundamentally the **integration by substitution** method is a specific method used to find **integrals** and is also sometimes termed as ***u*-substitution** or **the reverse chain rule**.

In this method, the integration is performed by first making a substitution, i.e., to **change** the **variable** and the **integrand**.

Typically, using the substitution method an unfamiliar integral is defined into the form that can be evaluated, i.e., substitution provides a simple integral involving the variable ***u***.

Basic Notion: When $u = f(x)$, then $du = f'(x)\, dx$

Substitution Rule

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

Where $u = g(x)$.

The **symbol '∫'** is termed as an **integral sign**.

Following are the five significant steps that are used in integration by substitution method.

Step 1: Select a new variable *u*

Step 2: Define the value *dx*

Step 3: Make the substitution

Step 4: Integrate resultant integral

Step 5: Return to the initial variable *x*

Let us understand the concept with the help of the following example.

Integration by Substituting $u = ax + b$

Example 1. Evaluate $\int (2x + 3)^4 \, dx$.

Solution: Follow the steps discussed above.

Step 1: Select a substitution function u

The substitution function is then $u = 2x + 3$

Step 2: Define the value dx

$$2x + 3 = u$$
$$(2x + 3)' \, dx = du$$
$$2 \, dx = du$$
$$dx = 1/2 \, du$$

Step 3: Make the substitution as follows

$$\int (2x + 3)^4 \, dx = \int u^4 \cdot \frac{1}{2} \cdot du$$

Step 4: Integrate resultant integral

$$\int (2x + 3)^4 \, dx = \int u^4 \cdot \frac{1}{2} \cdot du = \frac{1}{2} \int u^4 \, du = \frac{1}{2} \cdot \frac{u^{4+1}}{4+1} + C = \frac{u^5}{10} + C$$

Here 'C' is a constant for integration.

Step 5: Return to the initial variable x

$$\frac{u^5}{10} + C = \frac{(2x + 3)^5}{10} + C$$

Therefore, the answer is, $\int (2x + 3)^4 \, dx = \dfrac{(2x + 3)^5}{10} + C$

Now assume that we have to find the integral,

$$\int (x + 4)^5 \, dx \qquad \qquad \dots(1)$$

According to the **rule of integration** for the **powers of variable**, the **power** of a **variable** can be **increased by '1'** and then **divided** by the **new increased power**.

In the Equation (1), the **integral** has a **power '5'** whereas the **integrand** has a complex form as it has the **term '$x + 4$'**. To evaluate the integral, **substitution** is made, i.e., the integrand is changed into the simpler form as 'u^5'. Additionally, the term 'dx' too will be substituted appropriately. Hence, we take '$u = x + 4$'.

In the **differential term**, we can state that,

$$du = \left(\frac{du}{dx}\right) dx$$

Since, $u = x + 4$

Then, $(du/dx) = 1$

And thus, $du = dx$

Now on substituting the values for both '$x+4$' and 'dx' in the Equation (1), we obtain the equation of the form,

$$\int (x+4)^5 \, dx = \int u^5 du$$

The resultant integral is then evaluated which provides the equation of the form,

$$(u^6/6) + C \qquad\qquad \dots (2)$$

Where 'C' is a constant for integration.

Now we substitute the values of Equation (2) in the original variable 'x' in the Equation (1), i.e., '$u=x+4$' to have the following expression,

$$\int (x+4)^5 \, dx \;=\; ((x+4)^6/6) + C$$

This proves the method of **integration by substitution**.

6.2.2 Trigonometric Substitution

To evaluate an integral using trigonometric substitution let us first recollect the following standard trigonometric notations of trigonometric functions.

Trigonometric Functions

sin	=	Opposite/Hypothenuse	
cos	=	Adjacent/Hypothenuse	
tan	=	Opposite/Adjacent	= sin/cos
sec	=	Hypothenuse/Adjacent	= 1/sin
cosec	=	Hypothenuse/Opposite	= 1/cos
cot	=	Adjacent/Opposite	= 1/tan=cos/sin

Trigonometric Substitutions

Following are the standard integrands and notations for trigonometric substitutions. Since we are using trigonometric functions hence we will use θ **notation** instead of *u***-notation**.

Integrand	Substitution	Differential
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos \theta d\theta$
$\sqrt{x^2 + a^2}$	$x = a \tan \theta$	$dx = a \sec^2 \theta d\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec \theta \tan \theta d\theta$

When we substitute $x = a \sin \theta$, then $\sin \theta = x/a$.

Therefore, the opposite side to 'θ' of the triangle is labelled as 'x' while the hypothenuse is labelled as 'a', as shown in Figure 6.2. The remaining side, i.e., the adjacent, is defined on the basis of Pythagoras Theorem which states that,

$$a^2 + b^2 = c^2$$

Hence the adjacent will have the integrand of the form, $\sqrt{a^2 - x^2}$, as shown in Figure 6.2.

Similarly we define the substitutions for remaing two cases, i.e., for $\sqrt{x^2 + a^2}$ and $\sqrt{x^2 - a^2}$ as shown in Figure 6.2.
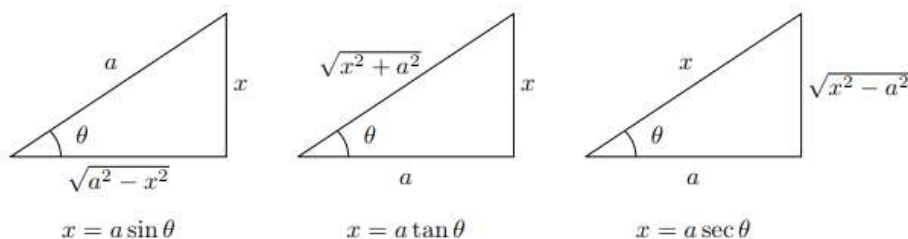
Fig. 6.2 Substitutions

Integrals Containing $\sqrt{x^2 + a^2}$

The integrals containing $\sqrt{x^2 + a^2}$ are integrated as follows. Basically, $\sqrt{x^2 + a^2}$ essentially correspond to the hypotenuse of the right angled triangle as shown in Figure 6.3.
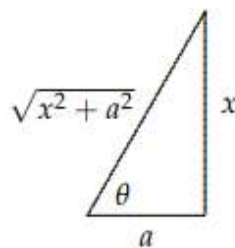
Fig. 6.3 Integrals Containing $\sqrt{x^2 + a^2}$

Hence, $x = a \tan \theta$

$dx = a \sec^2 \theta \, d\theta$

$\sqrt{x^2 + a^2} = a \sec \theta$

Example 2. Evaluate the indefinite integral of the form,

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

Solution: Follow the steps given below.

Use the triangle substitution we have,

$$a^2 = 4 \quad \text{or} \quad a = 2$$

Then integrating we have,

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta\,d\theta$$

$$= \tfrac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \tfrac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

$$= \tfrac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \tfrac{1}{4}\int (\sin\theta)^{-2}\cos\theta\,d\theta$$

$$= -\tfrac{1}{4}(\sin\theta)^{-1} + c$$

Now, we will convert the obtained answer to the function of '*x*'. Refer Figure 6.3 and define that,

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

Consequently,

$$-\tfrac{1}{4}(\sin\theta)^{-1} + c = -\tfrac{1}{4}\left(\frac{x}{\sqrt{x^2+4}}\right)^{-1}$$

$$= -\frac{\sqrt{x^2+4}}{4x} + c.$$

Integrals Containing $\sqrt{x^2-a^2}$

The integrals containing $\sqrt{x^2-a^2}$ are integrated as follows. Basically, '*x*' essentially correspond to the hypotenuse of the right angled triangle as shown in Figure 6.4.
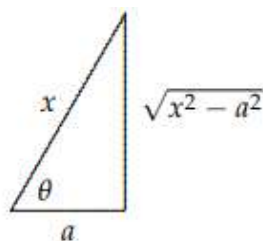
Fig. 6.4 *Integrals Containing* $\sqrt{x^2 - a^2}$

Hence,

$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

Example 3. Evaluate the indefinite integral of the form,

$$\int \frac{x^3}{\sqrt{x^2 - 16}} dx.$$

Solution: Follow the steps given below.

Since the integral $\int \frac{x^3}{\sqrt{x^2 - 16}} dx.$ contains the form $\sqrt{x^2 - 16}$, hence we will substitute as follows,

$$x = a \sec \theta = 4 \sec \theta$$

Therefore, we will have the triangle as shown in the following Figure 6.5.
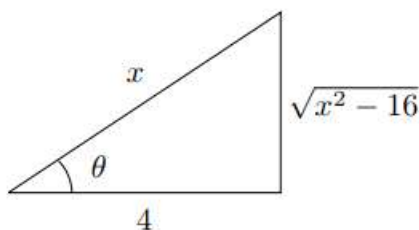
$$x = 4 \sec \theta$$

Fig. 6.5 *Triangle Showing Integrals*

Analysing this we have,

$$\sqrt{x^2 - 16}, = 4 \tan \theta$$

But, as per the standard formula,

$$dx = 4 \sec \theta \tan \theta \, d\theta$$

Subsequently, the integral will have the form,

$$\int \frac{x^3}{\sqrt{x^2-16}}\, dx = \int \frac{(4\sec\theta)^3}{4\tan\theta}(4\sec\theta\tan\theta)\, d\theta$$

$$= 64 \int \sec^4\theta\, d\theta$$

Applying the trigonometric integrals rules and the identity,

$$\sec^2 = \tan^2 + 1$$

The integral can be evaluated using the substitution,

$$u = \tan\theta$$

Therefore,

$$du = \sec^2\theta\, d\theta$$

Hence,

$$\int \frac{x^3}{\sqrt{x^2-16}}\, dx = 64 \int \sec^4\theta\, d\theta$$

$$= 64 \int (\tan^2\theta + 1)\sec^2\theta\, d\theta$$

$$= 64 \int (u^2 + 1)\, du$$

$$= 64\left(\frac{u^3}{3} + u\right) + C$$

$$= 64\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C$$

From the Figure 6.5 triangle it is obvious that,

$$\tan\theta = \sqrt{x^2-16}\,/\,4$$

Therefore, the integral will become,

$$\int \frac{x^3}{\sqrt{x^2-16}}\, dx = \frac{\left(\sqrt{x^2-16}\right)^3}{3} + 16\sqrt{x^2-16} + C$$

Integrals Containing $\sqrt{a^2-x^2}$

The integrals containing $\sqrt{a^2-x^2}$ are integrated as follows. Basically, '*a*' essentially correspond to the hypotenuse of the right angled triangle as shown in Figure 6.6. One side of the triangle is $\sqrt{a^2-x^2}$ and the other side will be '*x*'.
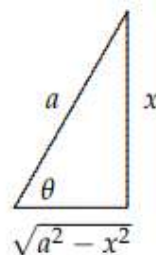
Figure 6.6 Integrals Containing $\sqrt{a^2 - x^2}$

Hence,

$$x = a \sin \theta$$

$$dx = a \cos \theta \, d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

Example 4. Evaluate the indefinite integral of the form,

$$\int \frac{4}{4 - x^2} \, dx.$$

Solution: Follow the steps given below.

We will consider the form $4 - x^2$ as, $(\sqrt{4 - x^2})^2$ to be used as integral of triangle as shown in Figure 6.7 .
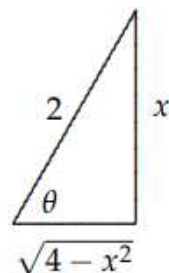
Fig. 6.7 *Triangle with Side* $\sqrt{4 - x^2}$

Therefore,

$$x = 2 \sin \theta$$

$$dx = 2 \cos \theta \, d\theta$$

$$\sqrt{4 - x^2} = 2 \cos \theta$$

The integral can be expressed as follows.

$$\int \frac{4}{4 - x^2} \, dx = \int \frac{4}{(2 \cos \theta)^2} \cdot 2 \cos \theta \, d\theta$$

$$= \int \frac{8 \cos \theta}{4 \cos^2 \theta} \, d\theta = 2 \int \sec \theta \, d\theta$$

$$= 2 \int \ln |\sec(\theta) + \tan \theta| \, d\theta$$

$$= 2 \ln \left| \frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}} \right| + c$$

$$= 2 \ln \left| \frac{2+x}{\sqrt{4-x^2}} \right| + c$$

$$= \ln \left| \frac{(2+x)^2}{4-x^2} \right| + c$$

$$= \ln |2 + x| - \ln |2 - x| + c.$$

Some More Significant Integration Formulas

The following formulas are specifically used in the integration by substitution method to simplify integrals with reference to trigonometric identities for a definite series of 'u' and in the approximation of 'du'.

    1. When an **integral** includes the form $[u^2 - a^2]^{1/2}$, then substitute for 'u' such that $u = a \operatorname{cosec} \theta$, consequently $[u^2 - a^2]^{1/2}$ has the form,

$$[a^2 \operatorname{cosec}^2 \theta - a^2]^{1/2} = [a^2 (\operatorname{cosec}^2 \theta - 1)]^{1/2}$$
$$= [a^2 \cot^2 \theta]^{1/2}$$
$$= a \cot \theta$$

Which follows the identity that $(du/d\theta)(a \operatorname{cosec} \theta) = - a \operatorname{cosec} \theta \cot \theta$

    2. When an **integral** includes the form $[u^2 - a^2]^{1/2}$, then substitute for 'u' such that $u = a \sec \theta$, consequently $[u^2 - a^2]^{1/2}$ has the form,

$$[a^2 \sec^2 \theta - a^2]^{1/2} = [a^2 (\sec^2 \theta - 1)]^{1/2}$$
$$= [a^2 \tan^2 \theta]^{1/2}$$
$$= a \tan \theta$$

Which follows the identity that $(du/d\theta)(a \sec \theta) = a \sec \theta \tan \theta$

    3. When an **integral** includes the form $[u^2 + a^2]^{1/2}$, then substitute for 'u' such that $u = a \tan \theta$, consequently $[u^2 + a^2]^{1/2}$ has the form,

$$[a^2 \tan^2 \theta + a^2]^{1/2} = [a^2 (\tan^2 \theta + 1)]^{1/2}$$
$$= [a^2 \sec^2 \theta]^{1/2}$$
$$= a \sec \theta$$
$$= a (1/(\cos^2 x))$$

Which follows the identity that $(du/d\theta)(a \tan \theta) = a \sec^2 \theta$

    4. When an **integral** includes the form $[a^2 - u^2]^{1/2}$, then substitute for 'u' such that $u = a \sin \theta$, consequently $[a^2 - u^2]^{1/2}$ has the form,

$$[a^2 - a^2 \sin^2 \theta]^{1/2} = [a^2 (1 - \sin^2 \theta)]^{1/2}$$
$$= [a^2 \cos^2 \theta]^{1/2}$$
$$= a \cos \theta$$

Which follows the identity that $(du/d\theta)\,(a\sin\theta)=a\cos\theta$

Example 5. Evaluate the indefinite integral of the form,

$$\int \frac{x^2+3x}{\sqrt{x+4}}\, dx$$

Solution: Follow the steps given below.

Consider that, $\qquad u = \sqrt{x+4}$

Hence, $\qquad du = \frac{1}{2}\,(x+4)^{-1/2}\,dx$

And, $\qquad x = u^2 - 4$

Subsequently,

$$\int \frac{x^2+3x}{\sqrt{x+4}}\, dx = \text{+"}\ 2\,((u^2-4)^2 + (u^2-4))\,du$$

$$= \text{+"}\ (2\,u^4 - 10\,u^2 + 8)\,du$$

$$= 2/5\,(u^5) - 10/3\,(u^3) + 8u + C$$

$$= 2/5\,(x+4)^{5/2} - 10/3\,(x+4)^{3/2} + 8\,(x+4)^{1/2} + C$$

Thus, the integral is evaluated using substitution.

Check Your Progress

1. What are indefinite integrals?

2. What are definite integrals?

3. Evalute $\int e^x\, dx$.

4. Evalute $\int 1/x\, dx$.

6.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. When it is not definite or specified how to start and end the process of integration then the integral is termed as 'indefinite integral'.

2. When it is definite or specific that how to start and end the process of integration, i.e., the upper and lower limits of integration (b and a) are defined then the integral is termed as 'definite integral'.

3. e^x.

4. $\log x$.

6.4 SUMMARY

- The integral of a function is approximated with regard to '*x*', i.e., estimating an area from the curve to the *x*-axis. Integration process is specifically used for finding or approximating central points, areas, volumes, etc.

- Integral of '*f*' from '*x* = *a*' to '*b*': $\int_a^b f(x)dx$.

- The integrals are of two types, indefinite and definite.

- Indefinite integral which is termed as the 'function of *x*'.

$$\int f(x)\, dx.$$

- The following is the definite integral which is termed as the 'number'.

$$\int_a^b f(x)\, dx$$

- Substitution Rule:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

- According to the rule of integration for the powers of variable, the power of a variable can be increased by '1' and then divided by the new increased power.

Some Significant Rules

Integrand	Substitution	Differential
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos \theta d\theta$
$\sqrt{x^2 + a^2}$	$x = a \tan \theta$	$dx = a \sec^2 \theta d\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec \theta \tan \theta d\theta$

6.5 KEYWORDS

- **Integral:** An integral is a mathematical object that can be interpreted as an area or a generalization of area.

- **Integration:** The finding of an integral or integrals.

- **Derivative:** The derivative is a way to show rate of change: that is, the amount by which a function is changing at one given point.

- **Differentiation:** The process of finding the derivative, or rate of change, of a function.

6.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Evaluate $\int x^2 + 2x - 1 \, dx$.
2. Evaluate $\int (x - 9)^2 \, dx$.
3. Evaluate $\int \sin 2x \, dx$.
4. Evaluate $\int \cos 2x \, dx$.

Long Answer Questions

1. Evaluate the indefinite integral of the form,

$$\int \frac{4}{1-x^2} \, dx$$

2. Evaluate the indefinite integral of the form,

$$\int \frac{2}{x^2+16} \, dx$$

3. Evaluate the indefinite integral of the form,

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$$

4. Evaluate the indefinite integral of the form,

$$\int \frac{x^2+5x+2}{\sqrt{x-1}} \, dx$$

5. Discuss the process of integration with help of examples for.

   (a) Integration by Substitution

   (b) Trigonometric Substitution

6.7 FURTHER READINGS

Arumugam, Issac. 2005. *Calculus*. Palayamkottai: New Gamma Publishing House.

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Sharma, A.K. 2004. *Text book of Differential Calculus*. New Delhi: Discovery publishing house.

Narayanan, S, T.K. Manickavasagam Pillai. 2009. *Differential Equations and its applications*. Chennai: S.Viswanathan(Printers & Publishers) Pvt. Ltd.

Venkataraman, M.K, Manorama Sridhar. *Calculus and Fourier series*. Chennai: The National Publishing Company.

UNIT 7   INTEGRATION – DEFINITE INTEGRALS

Structure

7.0 INTRODUCTION

In this unit, you will expand your knowledge on definite integral which is an integral of a function with limits of integration. There are two values as the limits for the interval of integration. One is the lower limit and the other is the upper limit. It does not contain any constant of integration. Further, you will learn about their properties. A very important method – integration by parts, is discussed in this unit. This is one of the most useful integration method which is applied in a variety of situations. It is useful for the integrals where the integrates are functions, any of which may be algebraic, exponential, trigonometric and logarithmic. This unit also introduces you to the concept of integration by reduction formula. Integration by reduction formula is a or procedure of integration, in the form of a recurrence relation. It is used when an expression containing an integer parameter, usually in the form of powers of elementary functions, or products of transcendental functions and polynomials of arbitrary degree, can't be integrated directly. In the end, this units discusses the notion of Bernoulli's formula.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of definite integrals
- Know about properties of definite integrals
- Learn the method of integratin by parts

- Deduce reduction formula for various functions
- Use Bernoulli's formula to integrate a function

7.2 INTEGRATION – DEFINITE INTEGRALS AND THEIR PROPERTIES

Definite integration is a significant and an essential constituent of integral calculus. Both the integrations – **indefinite** and **definite** are **interrelated** processes. Fundamentally, the indefinite integration provides the basis for **definite integral**. As already discussed in Unit 6, following is the standard definition of definite integral.

Definitions of Definite Integral

1. The **definite integral** of a **function** has a unique value. A definite integral is denoted by,

$$\int_a^b f(x)\,dx$$

In the above notation, '*a*' is termed as the **lower limit** of the integral while '*b*' is termed as the **upper limit** of the integral.

2. When it is definite or specific that how to start and end the process of integration, i.e., the **upper** and **lower limits** of **integration** (*b* and *a*) are defined then the **integral** is termed as **'definite integral'**. A definite integral is denoted by,

$$\int_a^b f(x)\,dx$$

3. Principally, the **definite integral** of the form $\int_a^b f(x)\,dx$ is evaluated either as,

    1. The **limit of the sum**.

      Or as,

    2. The **anti-derivative 'F'** for the **interval [*a*, *b*]**.

When we consider the case as anti-derivative 'F' for the interval [*a*, *b*], then its evaluated value is defined as the difference between the values of 'F' at the specified end points, to be precise **F(*b*) – F(*a*)**.

7.2.1 Evaluation of Definite Integral as the Limit of a Sum

Assume that '*f*' is a continuous function that is specified on close by interval [*a*, *b*]. Let the function has all non-negative values. In this case the graph of the function will be a curve above the *x*-axis.

The definite integral certainly will be $\int_a^b f(x)\, dx$, which is the specified area bounded by the curve '$y = f(x)$' with $x = a, x = b$ on the x-axis. For estimating the area between the curve, we evaluate the segment PRSQP along with, $x = a, x = b$ and the x-axis (Refer Figure 7.1).
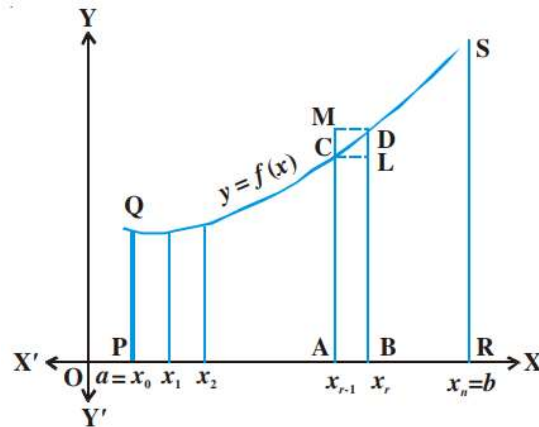
***Fig.* 7.1** *Area Between the Curve*

The interval $[a, b]$ is further divided into 'n' number of equivalent subintervals which are represented or denoted by,

$$[x_0, x_1], [x_1, x_2], \ldots\ldots\ldots, [x_{r-1}, x_r], \ldots\ldots\ldots, [x_{n-1}, x_n]$$

Here,

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \ldots\ldots, \quad x_r = a + rh$$

And,

$$x_n = b = a + nh \quad \text{or} \quad n = (b - a)/h$$

As per, $\qquad n \to \infty, \ h \to 0.$

The segment PRSQP is defined as the sum of n subsegments and subsegment is defined on the basis of subintervals $[x_{r-1}, x_r]$, where $r = 1, 2, 3, 4, \ldots, n$.

From Figure 7.1, we can state that,

ABLC (Area of the Rectangle) $<$ ABDCA (Area of the Segment) $<$ ABDM (Area of the Rectangle) …(1)

Evidently, since,

$$[x_r - x_{r-1}] \to 0, \quad \text{specifically as,} \ h \to 0$$

Consequently, with this condition all the three regions that are specified in the Equation (1) are considered almost equivalent to each other.

Accordingly, we can define the sum of intervals as follows.

$$s_n = h\,[f(x_0) + \dots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \dots(2)$$

And, $\quad S_n = h\,[f(x_1) + f(x_2) + \dots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \dots(3)$

The term 's_n' is used to represent the sum of the regions for all the lower rectangles while the term 'S_n' is used to represent the sum of the regions for all the upper rectangles considered over the subintervals, $[x_{r-1}, x_r]$, where $r = 1, 2, 3, 4,$ . . . . , n.

Now, taking into account the Equation (1) inequality we can define the following state for an arbitrary or random subinterval $[x_{r-1}, x_r]$,

$$s_n < \text{Area of the Segment PRSQP} < S_n \qquad \dots(4)$$

When '$n \to \infty$' turn out to be narrow then we assume that the limiting values of Equations (2) and (3) will be similar or equivalent in both the conditions. Hence, under this situation the limiting value which is common will be the required segment under the curve.

Symbolically, the common limiting value can be expressed as,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$$

$$= \text{Area of the Segment PRSQP}$$

$$= \int_a^b f(x)\,dx \qquad \dots(5)$$

The area specified under the segment PRSQP can also be defined as the limiting value of some other specific area which exists between the rectangles either below the stated curve or above the stated curve. We limit the condition and will consider only those rectangles which are of height equivalent to the stated curve and are at the left side edge of every subinterval.

Therefore, in limit form the Equation (5) can be expressed as,

$$\int_a^b f(x)\,dx = \lim_{h\to0} h\,[f(a) + f(a+h) + \dots + f(a + (n-1)\,h]$$

Or,

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \dots + f(a + (n-1)\,h]$$

$$\dots(6)$$

Here, $\quad h = \dfrac{b-a}{n} \to 0 \quad$ as $n \to \infty$.

Precisely, the **Equation (6)** is defined as the definition of the **definite integral** as the **'limit of sum'**.

Definition. The value of the definite integral of a function for some precise interval is specifically determined on the basis of function and the interval, but it will not consider the variable of integration selected for representing the independent variable.

Therefore, if the independent variable is symbolized through '*t*' or '*u*' in place of '*x*' then the integral of Equation (5),

$$= \int_a^b f(x)dx$$

Will be represented as,

$$\int_a^b f(t)\, dt \ \text{ or } \int_a^b f(u)\, du$$

Subsequently, the **variable of integration** will be termed as a **'dummy variable'**.

Example 1. Evaluate the given definite integral of the form as the limit of a sum,

$$\int_0^2 (x^2 + 1)\, dx$$

Solution: Follow the steps given below.

According to the definition,

$$\int_a^b f(x)\, dx = (b-a) \lim_{n \to \infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)h].$$

For $h = (b-a)/n$.

Hence we can state that,

$a = 0, \quad b = 2, \quad f(x) = x^2 + 1, \quad h = (2-0)/n = 2/n$

Therefore,

$$\int_0^2 (x^2+1)\, dx = 2 \lim_{n \to \infty} \frac{1}{n} [f(0) + f(\frac{2}{n}) + f(\frac{4}{n}) + \ldots + f(\frac{2(n-1)}{n})]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} [1 + (\frac{2^2}{n^2} + 1) + (\frac{4^2}{n^2} + 1) + \ldots + \left(\frac{(2n-2)^2}{n^2} + 1 \right)]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} [\underbrace{(1 + 1 + \ldots + 1)}_{n\text{-terms}} + \frac{1}{n^2} (2^2 + 4^2 + \ldots + (2n-2)^2]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} [n + \frac{2^2}{n^2} (1^2 + 2^2 + \ldots + (n-1)^2]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} [n + \frac{4}{n^2} \frac{(n-1)\, n\, (2n-1)}{6}]$$

$$= 2 \lim_{n \to \infty} \frac{1}{n} [n + \frac{2}{3} \frac{(n-1)\,(2n-1)}{n}]$$

$$= 2 \lim_{n \to \infty} [1 + \frac{2}{3}(1 - \frac{1}{n})\ (2 - \frac{1}{n})]$$

$$= 2\ [1 + (4/3)] = 14/3$$

7.2.2 Fundamental Theorems of Calculus – Area Function

The fundamental theorems of calculus defines how to evaluate the **area** of the **segment** that is bounded by the **curve** $y = f(x)$.

Area function can be defined with regard to the Equation (5),

$$= \int_a^b f(x) dx$$

It states the area of the segment that is bounded through the curve $y = f(x)$, the coordinates $x = a$ and $x = b$ on the x-axis. Consider that 'x' is a specific point in the interval $[a, b]$ as shown in Figure 7.2.

Then, $\int_a^x f(x)\ dx$ will specifically represent the area that is on the left-hand side of 'x' denoted as 'A(x)', as shown in Figure 7.2.
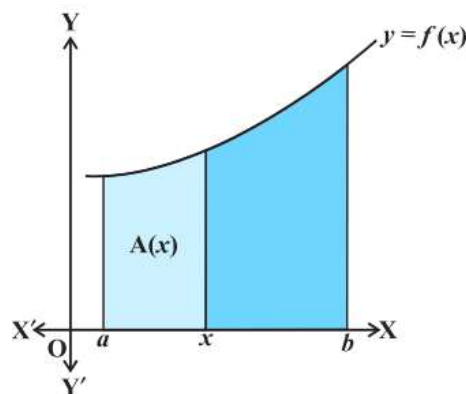
Fig. 7.2 Area Function

Consider that $f(x) > 0$ for $x \in [a, b]$. The area of the stated segment is determined on the value of 'x', i.e., the area of the stated segment is a function of 'x' denoted by A(x).

Therefore, the **function of 'x'** denoted by **A(x)** is termed as the **'Area Function'** and is represented as equation,

$$A(x) = \int_a^x f(x)\ dx \qquad \qquad …(7)$$

On the basis of Equation (7) the basic two fundamental theorems of integral calculus can be stated as given below.

Theorem 1. Let 'f' be a continuous function on the closed interval $[a, b]$ and let A(x) be the area function. Then,

$$A'(x) = f(x), \textbf{ for all } x \in [a, b]$$

Theorem 2. Let 'f' be a continuous function defined on the closed interval $[a, b]$ and 'F' be the anti-derivative of 'f'. Then,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Theorem 2 is considered as significant theorem of integral calculus as it helps in evaluating the definite integral using anti-derivative.

Example 2. Evaluate the given definite integral of the form,

$$\int_2^3 x^2\,dx$$

Solution: Follow the steps given below.

Consider that,

$$D = \int_2^3 x^2\,dx$$

Because,

$$\int x^2\,dx = \frac{x^3}{3} = F(x)$$

Hence, applying the Theorem 2 of integral calculus we obtain,

$$D = F(3) - F(2)$$
$$= (27/3) - (8/3)$$
$$= 19/3$$

Evaluation of Definite Integrals by Substitution Method

For evaluating the definite integrals using the substitution method follow the steps given below.

Step 1. To evaluate the definite integral of the form $= \int_a^b f(x)\,dx$ by the substitution method we first define it to the known indefinite integral form.

Step 2. Reduce the definite integral form to the indefinite integral form by substituting '$y = f(x)$' or '$x = g(y)$', i.e., the integer is without upper and lower limits.

Step 3. Now integrate the new obtained integrand with reference to new variable but no need to mention or use the constant of integration.

Step 4. Again substitute for the new variable to obtain the answer in the form of original expressions of the variable.

Step 5. Use this answer obtained in Step 4, to find the values for the specified limits of the integral, i.e., evaluate the difference of values at the specified upper limit and the lower limit.

Example 3. Evaluate the given definite integral of the form using substitution,

$$\int_{-1}^1 5x^4\sqrt{x^5 + 1}\,dx$$

Solution: Follow the steps given below.

Let, $t = x^5 + 1$ and then $dt = 5x^4\, dx$

Hence,

$$\int 5x^4 \sqrt{x^5 + 1}\, dx = \int \sqrt{t}\, dt$$

$$= \frac{2}{3} t^{\frac{3}{2}} = \frac{2}{3}(x^5 + 1)^{\frac{3}{2}}$$

Consequently,

$$\int_{-1}^{1} 5x^4 \sqrt{x^5 + 1}\, dx = \frac{2}{3}\left[(x^5 + 1)^{\frac{3}{2}} \right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5 + 1)^{\frac{3}{2}} - \left((-1)^5 + 1 \right)^{\frac{3}{2}} \right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right]$$

$$= \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

7.2.3 Properties of Definite Integrals

Following are some significant properties of definite integrals using which we can easily evaluate the definite integrals.

Property 1. $\displaystyle\int_{a}^{b} f(x)\, dx = \int_{a}^{b} f(t)\, dt$

To evaluate we substitute as '$x = t$'.

Property 2. $\displaystyle\int_{a}^{b} f(x)\, dx = -\int_{b}^{a} f(x)\, dx$

Specifically, $\displaystyle\int_{a}^{a} f(x)\, dx = 0$

Assume that 'F' be the anti-derivative of 'f'.

As per the Theorem 2 of integral calculus,

$$\int_{a}^{b} f(x)\, dx = F(b) - F(a)$$

$$= -[F(a) - F(b)]$$

$$= -\int_{b}^{a} f(x)\, dx$$

When $a = b$ then we can state that,

$$\int_{a}^{a} f(x)\, dx = 0$$

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Assuming that 'F' be the anti-derivative of 'f'. Then we can state,

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \dots(8)$$

$$\int_a^c f(x)\,dx = F(c) - F(a) \qquad \dots(9)$$

$$\int_c^b f(x)\,dx = F(b) - F(c) \qquad \dots(10)$$

We add Equations (9) and (10) to obtain,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$$

Hence the Property 3 is proved.

Property 4. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Consider that $t = a + b - x$. Thus, $dt = -dx$.

While $x = a$, $t = b$ and also when $x = b$, $t = a$.

We can state the definite integral as,

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \qquad \dots\text{By Property 2}$$

$$= \int_a^b f(a+b-x)\,dx \qquad \dots\text{By Property 1}$$

Property 5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Consider that, when $t = a - x$, then $dt = -dx$.

While $x = 0$, $t = a$ and also when $x = a$, $t = 0$.

The proof is similar to Property 4.

Property 6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Applying Property 3, we obtain,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Assume that,

$t = 2a - x$ for the right-hand side second integral.

Thus, $dt = -dx$.

While $x = a$, $t = a$ and also when $x = 2a$, $t = 0$ and precisely $x = 2a - t$.

Consequently, the right-hand side second integral has the form,

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\, dt$$

$$= \int_0^a f(2a-x)\, dx$$

Therefore,

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$$

Check Your Progress

1. How are definite integrals evaluated?

2. What does fundamental theorems of calculus define?

3. What are dummy variables?

7.3 INTEGRATION BY PARTS

In calculus and in mathematical analysis, **integration by parts** or **partial integration** method is used for finding the **integral** of a **product of functions** using expressions of the integral, specifically derivative and anti-derivative.

The rule is simply derived by integrating the product rule of differentiation.

Definitions

Following are the standrard definitions for the process integration by parts.

1. The **integration by parts** or **partial integration** process is specifically used to find the **integral** of a **product of functions**.

    If $\quad u = u(x) \quad$ and $\quad du = u'(x)\, dx$

    When $\quad v = v(x) \quad$ and $\quad dv = v'(x)\, dx$

    Then integration by parts states that,

    $$\int_a^b u(x)v'(x)\, dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\, dx$$

    $$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\, dx$$

    Or more precisely,

    $$\int u\, dv = uv - \int v\, du.$$

2. Sometimes an **integration** is the **product** of '**2**' **functions** which can be **integrated** using **Integration by Parts**.

    If u and v are functions of x, then using the product rule for differentiation we obtain the following equation,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Reorganizing the equation we have,

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating with reference to '*x*' the following formula is obtained for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Precisely, we use the function '*u*' for the reason that '*du/dx*' can be simple than '*u*'. When in an equation the expression contains mix functions that are to be integrated then we can define these mix functions as '*u*' to make the integration process easy, for example we can define '*u*' as,

$$u = \ln x, \qquad u = x^n, \qquad u = e^{nx}, \quad \text{etc.}$$

3. If '***u***' and '***v***' are any **two differentiable functions** of a **single variable** '***x***' then through the **product rule of differentiation**,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

When both the sides are integrated then we obtain,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Alternatively, $\quad \int u\dfrac{dv}{dx}\,dx = uv - \int v\dfrac{du}{dx}\,dx \qquad \qquad \text{…(11)}$

Consider that,

$$u = f(x) \quad \text{and} \quad dv/dx = g(x)$$

Then,

$$du/dx = f'(x) \quad \text{and} \quad v = +\text{''}g(x)\,dx$$

Consequently, we can express Equation (11) as follows,

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int [\int g(x)\,dx]\,f'(x)\,dx$$

Specifically,

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int [f'(x)\int g(x)\,dx]\,dx$$

Consider that if first function is '*f*' and the second function is '*g*' then we can state,

Integral of the Product of Two Functions = [First Function] × [Integral of Second Function] – Integral of [(Differential Coefficient of First Function) × (Integral of Second Function)]

Remember that integration by parts as product of functions can be applied only to specific cases and is not applicable to all.

For example, integration by parts as product of functions will not evaluate the expression of the form $\int \sqrt{x} \sin x \, dx$ because there is not any function which has required derivative as $\sqrt{x} \sin x$.

Example 4. Evaluate the given integral using integration by parts,

$$\int x \, \sin 2x \, dx$$

Solution: Follow the steps given below.

Consider that $u = x$ and $dv = \sin 2x \, dx$.

Since $u = x$ then $du = dx$.

On integrating $dv = \sin 2x \, dx$ gives,

$$v = \int \sin 2x \, dx$$

$$= \frac{-\cos 2x}{2}$$

Now we substitute these expressions into the formula of integration by parts as follows.

$$\int u \, dv = uv - \int v \, du$$

Substitute the following in the above expression,

$$u = x, \quad dv = \sin 2x \, dx, \quad du = dx, \quad v = \frac{-\cos 2x}{2}$$

We obtain,

$$\int \underbrace{u}_{} \underbrace{dv}_{} = \underbrace{u}_{} \underbrace{v}_{} - \int \underbrace{v}_{} \underbrace{du}_{}$$

$$\int \boxed{x} \, \boxed{\sin 2x \, dx} = \boxed{x} \, \boxed{\frac{-\cos 2x}{2}} - \int \boxed{\frac{-\cos 2x}{2}} \, \boxed{dx}$$

Therefore,

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \frac{\sin 2x}{2} + K$$

$$= \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} + K$$

Where K is a constant.

7.4 REDUCTION FORMULAE

In integral calculus, the **integration by reduction formula** is a specific technique
or method of integration of the form **recurrence relation**.

Typically, a **reduction formula** is used to solve an integral by reducing it to
an easier integral form and then further reducing it to the more easier form, and so
on.

Definition. A **reduction formula** for a **specified integral** is an **integral**
of the form which is **equivalent** in form as the specified integral but of a **lower
degree** or **order**.

Characteristically, the reduction formula is used for evaluating those specified
integrals which cannot be approximated otherwise. The specified integral is
evaluated by repeatedly using the reduction formula.

Definition. For some **specified integrals**, both **definite** and **indefinite**,
typically the **function** to be **integrated**, i.e., the **'integrand'** comprises of a product
of two functions, where one function contains an **unspecified** or **indefinite integer**
'n'. To evaluate such a specified integral in expressions of similar analogous integral
we use the method **integration by parts** where **'n'** is replaced by either **(n – 1)**
or at times by **(n – 2)**. The **correlation** between these **two integrals** is termed as
'reduction formula'. We can determine the **original integral** in terms of **'n'** by
repeatedly using this reduction formula.

Definition. Reduction formulae are certain **integrals** which contain some
variable 'n' in addition to the standard **variable 'x'**, generally obtained by means
of **integration by parts**. The notation I_n is used for denoting reduction formulae.

Following are some commonly used significant reduction formulae for
trigonometric integrals.

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

$$\int \csc^n(x)\,dx = -\frac{1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \cot^n(x)\,dx = \frac{-1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx$$

Specifically, we can state that,

Reduction Formula for Sine

$$\int \sin^n x \, dx = -\tfrac{1}{n} \cos x \sin^{n-1} x + \tfrac{n-1}{n} \int \sin^{n-2} x \, dx.$$

Reduction Formula for Cosine

$$\int \cos^n x \, dx = \tfrac{1}{n} \sin x \cos^{n-1} x + \tfrac{n-1}{n} \int \cos^{n-2} x \, dx.$$

Reduction Formula for Natural Logarithm

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx.$$

Let us understand the concept of reduction formula with the help of following example in which an integral is solved by reducing it to the form of more easy integral.

Let,

$$I_n = \int x^n e^x \, dx$$

This expression is simplified with regard to integration by parts as follows.

$$I_n = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$I_n = x^n e^x - n I_{n-1}$$

Thus, we obtain the above reduction formula.

Similarly, when,

$$I_n = \int \sec^n (\theta) d\theta$$

Again this expression is also simplified with regard to integration by parts as follows.

$$I_n = \sec^{n-2}(\theta) \tan(\theta) - (n-2) \int \sec^{n-2}(\theta) \tan^2(\theta) d\theta$$

Applying the trigonometric identity **tan² (θ) = sec² (θ) – 1**, we obtain,

$$I_n = \sec^{n-2}(\theta) \tan(\theta) + (n-2) \left(\int \sec^{n-2}(\theta)d\theta - \int \sec^n(\theta)d\theta \right)$$

$$= \sec^{n-2}(\theta) \tan(\theta) + (n-2)(I_{n-2} - I_n)$$

Rearrange or reorganize to obtain,

$$I_n = \frac{\sec^{n-2}(\theta) \tan(\theta)}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

When we obtain $n = 1$ or $n = 2$ then we stop integrating, where 'n' is either odd or even, respectively.

Example 5. Evaluate the reduction formula for the given integral of the form,

$$\int \left(x^2 + 1\right)^n dx \quad \text{(Where } n \text{ is a constant)}$$

Solution: Follow the steps given below.

Use the form $\int u\,dv$.

Select $u = (x^2 + 1)^n$ and $dv = dx$

Then $du = n\,(x^2 + 1)^{n-1}\,2x$ and $v = x$

Therefore,

$$\int \left(x^2 + 1\right)^n dx = \left(x^2 + 1\right)^n x - \int x \cdot n \left(x^2 + 1\right)^{n-1} 2x\,dx$$

$$= x \left(x^2 + 1\right)^n - 2n \int x^2 \left(x^2 + 1\right)^{n-1} dx$$

$$= x \left(x^2 + 1\right)^n - 2n \int \left[\left(x^2 + 1\right)^n - \left(x^2 + 1\right)^{n-1}\right] dx$$

$$= x \left(x^2 + 1\right)^n - 2n \int \left(x^2 + 1\right)^n dx + 2n \int \left(x^2 + 1\right)^{n-1} dx$$

Rearrange the expression to obtain,

$$(2n + 1) \int \left(x^2 + 1\right)^n dx = x \left(x^2 + 1\right)^n - 2n \int \left(x^2 + 1\right)^{n-1} dx.$$

Therefore, we obtain the following two recursive formulae.

$$\int \left(x^2 + 1\right)^n dx = \frac{x \left(x^2 + 1\right)^n}{2n + 1} - \frac{2n}{2n + 1} \int \left(x^2 + 1\right)^{n-1} dx \quad \left(n \neq -\frac{1}{2}\right)$$

And,

$$\int \left(x^2 + 1\right)^{n-1} dx = \frac{x \left(x^2 + 1\right)^n}{2n} - \frac{2n + 1}{2n} \int \left(x^2 + 1\right)^n dx \quad (n \neq 0).$$

Here Formula 1 is for positive 'n' while Formula 2 is for negative 'n'.

7.5 BERNOULLI'S FORMULA

Bernoulli equation is termed as the **nonlinear differential equations** of the **first order** and is represented as,

$$y' + a(x)\,y = b(x)\,y^m$$

Here $a(x)$ and $b(x)$ are termed as the **continuous functions**.

When $m = 0$, then the equation is defined as **linear differential equation** while when $m = 1$ then the equation is defined as **separable**.

Generally, when $m \neq 0, 1$ then the Bernoulli equation is converted to **linear differential equation** using the method **change of variable**,

$$z = y^{1-m}$$

The **function $z(x)$** will be defined as **differential equation** of the form,

$$z' + (1 - m)\, a\,(x)\, z = (1 - m)\, b\,(x)$$

Example 6. Evaluate the general solution of the equation $y' - y = y^2\, e^x$.

Solution: Follow the steps given below.

Assume that $m = 2$ for the specified Bernoulli equation and then apply the substitution as follows.

$$z = y^{1-m} = 1/y$$

Differentiating the equations on both sides, considering 'y' in the right-hand side expression as composite function of 'x', we have

$$z' = \left(\frac{1}{y}\right)' = -\frac{1}{y^2}\, y'.$$

Now the original differential equation is divided both sides by y^2, we have,

$$y' - y = y^2 e^x, \quad \Rightarrow \frac{y'}{y^2} - \frac{1}{y} = e^x.$$

Substituting z and z', we obtain,

$$-z - z = e^x$$

$$\Rightarrow \quad z' + z = -e^x$$

Then obtain the linear equation for the specified function $z(x)$. For solving we apply the integrating factor as follows,

$$u\,(x) = e^{\int 1 dx} = e^x.$$

Consequently, the linear equation has the general solution as follows,

$$z\,(x) = \frac{\int u\,(x)\, f\,(x)\, dx + C}{u\,(x)} = \frac{\int e^x\,(-e^x)\, dx + C}{e^x}$$

$$= \frac{-x + C}{e^x}$$

$$= (C - x)\, e^{-x}.$$

When we return to the function $y(x)$, we have the following implicit expression of the form,

$$y = \frac{1}{z} = \frac{1}{(C - x)\, e^{-x}};$$

This can be expressed as,

$$y\,(C - x) = e^x$$

Therefore,

$$y\,(C-x)=e^x, \quad y=0$$

Check Your Progress

4. Define integration by parts.

5. What is the reduction formula for Sine?

6. What is the reduction formula for natural logarithm?

7.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The definite integral of the form $\int_a^b f(x)\,dx$ is evaluated either as,

   • The **limit of the sum**. Or as,

   • The **anti-derivative 'F'** for the **interval [a, b]**.

2. The fundamental theorems of calculus defines how to evaluate the area of the segment that is bounded by the curve $y = f(x)$.

3. The variable of integration will be termed as a 'dummy variable'.

4. The integration by parts or partial integration process is specifically used to find the integral of a product of functions. If $u = u(x)$ and $du = u'(x)\,dx$, when $v = v(x)$ and $dv = v'(x)\,dx$. Then integration by parts states that,

$$\int u\,dv = uv - \int v\,du.$$

5. $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

6. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$

7.7 SUMMARY

• The **definite integral** of a **function** has a unique value. A definite integral is denoted by,

$$\int_a^b f(x)\,dx$$

• Let 'f' be a continuous function on the closed interval $[a, b]$ and let A(x) be the area function. Then, **A$'(x) = f(x)$, for all $x \in [a, b]$**

• Let 'f' be a continuous function defined on the closed interval $[a, b]$ and 'F' be the anti-derivative of 'f'. Then,

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a).$$

- $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

- $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

- $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

- $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

- Integral of the product of two functions = [first function] x [integral of second function] – integral of [(differential coefficient of first function) x (integral of second function)].

- A reduction formula for a specified integral is an integral of the form which is equivalent in form as the specified integral but of a lower degree or order.

- Bernoulli equation is termed as the nonlinear differential equations of the first order and is represented as, $y' + a(x)\, y = b(x)\, y^m$. Here $a(x)$ and $b(x)$ are termed as the continuous functions.

7.8 KEY WORDS

- **Continuous function:** a continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.

- **Class interval:** the size of each class into which a range of a variable is divided.

- **Anti-derivative**: an antiderivative or indefinite integral of a function f is a differentiable function F whose derivative is equal to the original function f.

7.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What do you understand by definite integrals?

2. Evaluate the given definite integral using anti-derivative,

   $\int_2^4 10x^2\, dx$

3. Prove that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. Define Bernoulli's Formula.

Long Answer Questions

1. Explain the process of evaluating definite integral as the limit of a sum.
2. Evaluate the given definite integral of the form as the limit of a sum,

   $\int_2^4 5x + 10\,dx$

3. Evaluate the given definite integral of the form as the limit of a sum,

   $\int_0^1 (x-9)^2\,dx$

4. Evaluate the given definite integral of the form using substitution,

   $\int_0^1 2x(x^2+1)\,dx$

5. Evaluate the reduction formula for the given integral of the form,

   $\int (x+1)\,dx$

7.10 FURTHER READINGS

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UNIT 8   DOUBLE AND TRIPLE INTEGRALS

Structure

8.0   INTRODUCTION

As discussed in Units 6 and 7 that 'Definite integration' is an essential and important constituent of integral calculus in which both the integrations – indefinite and definite are interrelated processes. You have learned about the single integrals and how to evaluate the function of one variable. In this unit, you will learn about double and triple integrals or the multiple integral, i.e., about two and three variables, respectively.

8.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept double integrals

- Describe the properties of double integrals

- Understand the concept triple integrals

- Describe the properties of triple integrals

- Use Jacobian transformation for changing variables in both the double and triple integrals.

- Change the order of integration

8.2 DOUBLE AND TRIPLE INTEGRALS AND THEIR PROPERTIES

Definition of Double and Triple Integrals

1. Fundamentally, the **multiple integral** is defined as a **'definite integral'** of a **function** of **more than one real variable**, such as $f(x, y)$ or $f(x, y, z)$. **Integrals** of a **function** of **'two variables'** over a **region** in R^2 are termed as **'double integrals'** while the **integrals** of a **function** of **'three variables'** over a **region** in R^3 are termed as **'triple integrals'**.

As already discussed in previous Unit 7 that the definite integral of single (one) variable of the positive function denotes the area of the specific region between the graph of the stated function and the x-axis.

2. Similarly, the **definite integral** of **two variables** of the **positive function** denotes the **volume** of the **specific region** between the surface that is defined by the specified function on the **three-dimensional Cartesian plane** when $z = f(x, y)$ in addition to the plane that holds its **domain**.

3. **Multiple integration** of a **function** in **'n' variables**, such as $f(x_1, x_2, \ldots, x_n)$ over a **'domain D'** is generally represented or denoted by **nested integral signs** in the **reverse order** of **execution** followed by the **function** and the **integrand arguments** in appropriate order. The **domain of integration** is either denoted symbolically for every argument over each integral sign or it is abbreviated by a variable at the rightmost integral sign as follows,

$$\int \cdots \int_{\mathbf{D}} f(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n$$

8.2.1 Double Integrals

Double integrals are the integrals using which we can find the **function** of **two variables**. Following are some standard definitions of double integral.

Definitions.

1. A **double integral** has the form,

$$\iint_R f(x, y)\, dx\, dy$$

Here **'R'** is termed as the **region of integration**, precisely the **region** in the (x, y) **plane**. Fundamentally, the double integral evaluates the **volume** under the specified **surface** $z = f(x, y)$.

2. The **double integral** that defines the limit of Riemann sums and which approximates the **volume** under the **graph of** $f(x, y)$ over the **planar region** R has the form,

$$\iint_R f(x, y)\, dA$$

3. The **double integrals** are the **iterated (repeated) integrals** of the form,

$$\iint f(x, y)\, dA = \iint f(x, y)\, dx\, dy = \int\left[\int f(x, y)\, dx\right] dy$$

For integrating over a **specified rectangular region (rectangle a, b, c, d)** on the **plane (x, y)** precisely define that $\int_a^b \int_c^d f(x, y)\, dx\, dy$ integrates 'f' over the stated **rectangle $a \leq y \leq b$, $c \leq x \leq d$**.

4. For a logically well-mannered **function $f(x, y)$** the **volume** can be calculated by taking the **limit** of these approximations termed as the **double integral**. The double integral of the function $f(x, y)$ to be the limit over all possible or potential divisions of **domain 'D'** can be defined as follows,

$$\iint_D f(x, y)\, dA = \lim_{A \to 0} \sum_{i=1}^n f(x_i^*, y_i^*) A_i$$

Consider that **domain D** is in $\mathbb{R}^2$. For the certain assumed division of D into 'n' specified regions state that in the above notation,

- A_i = The area of the ith region.
- (x_i^*, y_i^*) = Any point in the ith region.
- A = Area of the prime region.

Integral Laws for Double Integrals

The following integrals laws are applicable to double integrals only if the integrals exist.

1. Sum Rule

$$\iint_D f(x, y) + g(x, y)\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

2. Constant Multiple Rule

$$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA$$

3. Specific Rule: When the **domain D** is the **union** of **two subdomains D_1** and D_2 which are non-overlapping, then we use the notation,

$$\iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA$$

Fubini Theorem

The **Fubini Theorem** states that, "If $f(x, y)$ is **continuous** on a **region R**,

$R = \{(x, y) : a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x)\}$

$R = \{(x, y) : c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y)\}$"

Then,

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Iterated Integrals

The **iterated** means **repeated integrals**. The functions of various different variables can be integrated, for instance when the specified **partial derivative** is of the form,

$$f_x(x, y) = 2xy$$

then consider 'y' as **constant** and **integrate** with regard to 'x'. On integrating we have,

Integrating with regard to x,

$$f(x, y) = \int f_x(x, y)\, dx$$

Holding y as **constant**,

$$= \int 2xy\, dx$$

Constant y is factored,

$$= y \int 2x\, dx$$

Since **anti-derivative** of $2x$ is x^2 hence,

$$= y(x^2) + C(y)$$

Now $C(y)$ being **function** of y we have,

$$= x^2 y + C(y)$$

Here, $C(y)$ being **function** of y is termed as '**constant of integration**'.

In the same way, we can hold '*x*' as constant and integrate with regard to '*y*'. Both the notations are represented as follows,

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\, dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \qquad \text{With respect to } x$$

$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\, dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \qquad \text{With respect to } y$$

Evaluation of Double Integrals

The evaluation of double integral is done in stages as stated below. To evaluate the simplest form of **region R** for a **specified rectangular surface** is given as,

$$\iint_R f(x, y)\, dx\, dy$$

1. Estimate the limits of integration if not specified.
2. Estimate the inner integral for characteristic '*y*'.
3. Estimate the outer integral.

Double Integrals over General Regions: To evaluate the double integral over specified regions follow the steps given below.

1. Set the double integrals for the regions which are not of rectangular shape.
2. Evaluate the integrals for the bounds or limits containing variables.
3. Define +e*dy* as the outer integral.
4. Evaluate the change of bounds or limits.

Evaluating Double Integrals over General Regions

The double integral can be evaluated for the following two precise regions.

Type 1 Region *R*: Consider that the **Region *R*** is bounded by $x = a$, $x = b$, $y = p(x)$ and $y = q(x)$ when $a < b$ and $p(x) < q(x)$ for all $x \in [a, b]$ subsequently the **double integral** with regard to **Fubini Theorem** is represented as,

$$\iint_R f(x, y)\, dx\, dy = \int_{x=a}^{x=b} \int_{y=p(x)}^{y=q(x)} f(x, y)\, dy\, dx$$

Type 2 Region *R*: Consider that the **Region *R*** is bounded through the graphs of the specified **functions** $x = u(y)$, $x = v(y)$, $y = c$, $y = d$ on condition that $c < d$ and $u(y) < v(y)$ for all $y \in [c, d]$ subsequently the **double integral** over the **region '*R*'** is stated using the **iterated integral** with regard to **Fubini Theorem** is represented as,

$$\iint\limits_{R} f(x,y) \, dx \, dy = \int\limits_{y=c}^{y=d} \int\limits_{x=u(y)}^{x=v(y)} f(x,y) \, dx \, dy.$$

Example 1. Evaluate the given double integral of the form,

$$\int_{y=1}^{2} \int_{x=0}^{3} (1 + 8xy) \, dx \, dy$$

Solution: Follow the steps given below.

Here the given inner integral when 'y' is considered as constant is given as,

$$\int_{x=0}^{3} (1 + 8xy) \, dx$$

Integral (Considering 'y' as Constant),

$$= \int_{y=1}^{2} \left(\underbrace{\int_{x=0}^{3} (1 + 8xy) \, dx}_{\text{Considering '}y\text{' as Constant}} \right) dy$$

$$= \int_{y=1}^{2} \left[x + \frac{8x^2 y}{2} \right]_{x=0}^{3} dy$$

$$= \int_{y=1}^{2} (3 + 36y) \, dy$$

$$= \left[3y + \frac{36y^2}{2} \right]_{y=1}^{2}$$

$$= (6 + 72) - (3 + 18) = 57$$

Example 2. Evaluate the given double integral of the form,

$$\int_{0}^{\pi/2} \int_{0}^{1} y \sin x \, dy \, dx$$

Solution: Follow the steps given below.

The given integral is,

$$= \int_{0}^{\pi/2} \left(\int_{0}^{1} y \sin x \, dy \right) dx$$

$$= \int_0^{\pi/2} \left[\frac{y^2}{2} \sin x \right]_{y=0}^{1} dx$$

$$= \int_0^{\pi/2} \tfrac{1}{2} \sin x \, dx$$

$$= \left[-\tfrac{1}{2} \cos x \right]_{x=0}^{\pi/2} \quad = \tfrac{1}{2}$$

Evaluating Double Integral for Area and Volume

To evaluate the area under double integral follow the 'Area Definition' that is given below.

Definition for Area. Given a **function** $f: [a, b] \subseteq R \to R$ of any single **variable** '*x*' which is continuous and nonnegative on a **closed bounded interval [a, b]** on the *x*-axis. To evaluate the area of the plane region that is enclosed between the **curve** $y = f(x)$ and the **interval [a, b]**, we use the notation,

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{N} f(x_k^*) \Delta x_k = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*) \Delta x_k \qquad \ldots(1)$$

In the above Equation (1) the expression on the extreme right-hand side uses the **'limit as $N \to \infty$'** for encapsulating the procedure through the number of **intervals of [a, b]** are increased such that the **lengths of the subintervals** approaches **'0'**.

To evaluate the volume under double integral follow the 'Definition – Volume Under a Surface' that is given below.

Definition – Volume Under a Surface. When a **function** $f: D \subseteq R^2 \to R$ of any **two variables** '*x*' and '*y*' is defined such that '*f*' is continuous and nonnegative on the specific **region '*D*'** on the *xy*-**plane**, then the **volume** of the portion say '*E*' that is enclosed or bounded between the **surface** $z = f(x, y)$ and the **region '*D*'** is defined by the notation,

$$V(E) = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta A_k \qquad \ldots(2)$$

In the Equation (2) the expression on the right-hand side uses **'limit as $N \to \infty$'** which indicates or specifies the method or procedure through which the **number of subrectangles** of the specified **rectangle '*R*'** that encloses the **region '*D*'** are increased such that both the **length** and the **width of the subrectangles** approaches **'0'**.

As per the definition, **'ƒ'** is defined as **nonnegative** on the specified **region 'D'**. When **'ƒ'** is defined as **continuous** on the specified **region 'D'** and includes both **positive** and **negative values** then the **limit** is represented as,

$$\lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, x_k^*) \, \Delta A_k \qquad \qquad \ldots(3)$$

The **limit** defined in the Equation (3) will then **not represent volume** between the specified **region 'D'** and the **surface $z = f(x, y)$**, relatively it will specify the **'difference of volumes'**, i.e., the **volume** between **region 'D'** and the **portion** or **slice** of the **surface** that is below the **xy-plane**. This is termed as the **'net signed volume'** between the specified **region 'D'** and the **surface $z = f(x, y)$**.

The **double integral** of the **function $f(x, y)$** over the **region 'D'** is expressed as the **limit** of the **Riemann Sums** (if exists), which is obtained by extension as the sums in Equation (3) termed as the Riemann sums, and is represented by the equation of the form,

$$\int \int_D f(x, y) \, dA = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta A_k \qquad \qquad \ldots(4)$$

When **'ƒ'** is continuous and nonnegative on the **region 'D'** then the formula for **Volume of the portion E** or **V(E)** expressed in the Equation (2) will be denoted as follows,

$$V(E) = \int \int_D f(x, y) \, dA \qquad \qquad \ldots(5)$$

Definition. When **'ƒ'** contains both **positive** and **negative values** on the specified **region 'D'**, then the **positive value** states that **volume** is **more above** the **region 'D'** as compared to **below** of the region 'D' for the **double integral** of **'ƒ'** over the **region 'D'**, while the the **negative value** states that **volume** is **more below** the **region 'D'** as compared to **above** of the region 'D' for the **double integral** of **'ƒ'** over the **region 'D'**, and the **value '0' states** that for the **double integral** of **'ƒ'** over the **region 'D'** the **volume above** and **below** the **region 'D'** will be **equivalent** or same.

This holds the following theorem.

Theorem. When $f(x, y)$ is considered **continuous** in the **bounded region 'D'** then the following notation always exist,

$$\int \int_D f(x, y) dx dy \qquad \qquad \ldots(6)$$

For example, when $f(x, y) \geq 0$ in the specified **region 'D'** then we can interpret the **double integral** as the **volume of the cylinder** (Refer Figure 8.1) between the **surface** $z = f(x, y)$ and the **specified region 'D'**.
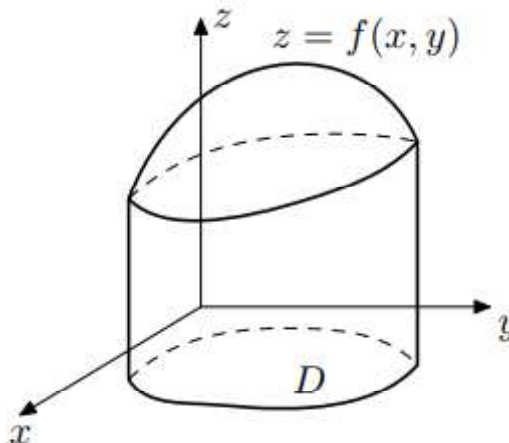
Fig. 8.1 *Cylinder Between the Surface z = f(x, y) and the Specified Region 'D'*

Alternatively, we can also define the **volume under** the specified **surface** $z = (x, y)$ above the stated **rectangular region 'R'** is represented as follows,

$$V = \iint_R f(x, y) \, dx \, dy \qquad \qquad \ldots(7)$$

Example 3. Evaluate the volume of the specified solid bounded region above with regard to the plane $z = 4 - x - y$ and below with regard to the rectangle R where,

$$R = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2\}$$

Solution: Follow the steps given below.

As stated above in the Equation (7), we can define the volume under some specified surface $z = (x, y)$ above the specified rectangular region 'R' as follows,

$$V = \iint_R f(x, y) \, dx \, dy$$

Given is, $\qquad \qquad z = 4 - x - y$, i.e., $f(x, y) = 4 - x - y$

For, $\qquad \qquad 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2$

Putting the values we obtain the double integral for volume as follows,

$$V = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

On integrating,

$$= \int_0^2 \left[4x - \tfrac{1}{2}x^2 - yx \right]_{x=0}^1 dy$$

$$= \int_0^2 (4 - \tfrac{1}{2} - y)\, dy$$

$$= \left[\frac{7y}{2} - \frac{y^2}{2} \right]_{y=0}^2$$

$$= (7 - 2) - (0) = 5$$

In this example of double integrals all the specified four limits of integration are considered as constant because the shape of the region of integration is rectangle.

Example 4. Evaluate the area of the rectangular region R given below in Figure 8.2 using an iterated integral.
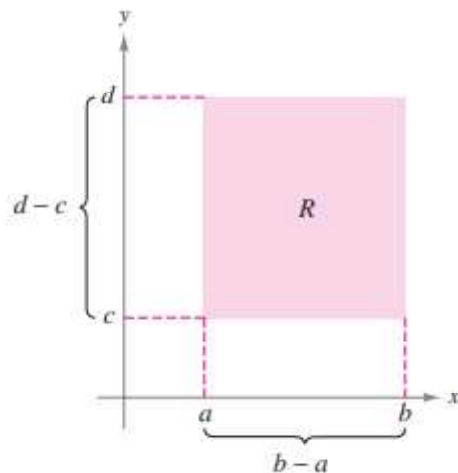
Fig. 8.1. *Area for Rectangular Region*

Solution: Follow the steps given below.

Figure 8.1 represents the region which is simple both vertically and horizontally, therefore we will use the order of integration by selecting the order '*dy dx*' for obtaining the given equations.

$$\int_a^b \int_c^d dy\, dx = \int_a^b y \Big]_c^d dx$$

Integrating with regard to '*y*' we obtain,

$$= \int_a^b (d - c)\, dx$$

Integrating with regard to '*x*' we obtain,

$$= \left[(d - c)x \right]_a^b$$

Area of the Rectangular Region *R* is,
$$= (d - c)\,(b - a)$$

8.2.2 Triple Integrals

In this section we will discuss about triple integral and the method of evaluating the triple integrals. Fundamentally, the **triple integrals** are defined as the **integrals over** a **three-dimensional region**.

Definitions of Triple Integral

1. Integration of a **function** of **three variables**, say $w = f(x, y, z)$ over a **three-dimensional region *R*** in ***xyz*-space** is termed a **triple integral** and is denoted as follows,

$$\iiint_R f(x, y, z)\, dV$$

2. If '***f***' is **continuous** over a **bounded solid region '*Q*'**, then the **triple integral** of '***f***' **over '*Q*'** is defined as follows provided that the limit exists,

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i \qquad \ldots(8)$$

The **volume** of the **solid region '*Q*'** is denoted as follows,

$$\text{Volume of } Q = \iiint_Q dV \qquad \ldots(9)$$

3. Consider that a **domain '*D*'** is specified in the **three dimensional space** and a **function** $f(x, y, z)$, then the **region '*D*'** can be **subdivided** into following **regions**:

- V_i = Volume of the *i*th region.
- (x_i^*, y_i^*, z_i^*) = Points in the *i*th region.
- V = Volume of the largest region.

The **triple integral** of '*f*' is then defined **over '*D*'** with regard to the following given **limit over all** the **possible divisions** of '*D*':

$$\iiint_D f(x, y, z)dV = \lim_{V \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)V_i \qquad \ldots(10)$$

4. Triple Integral according to Fubini's Theorem: As per the **Fubini's Theorem** the **triple integral** is evaluated using the equation of the form,

$$\iiint_D f(x, y, z)dV = \iiint_D f(x, y, z)dxdydz \qquad \ldots(11)$$

Here *dx*, *dy* and *dz* can follow **any order** for **integration**.

Fundamentally, the **value** of the **triple integral** may depend on the **order of integration** but the **Fubini's Theorem** ensures that the **order of integration** will not be significant when the **function is continuous**.

Consequently, as per the **Fubini's Theorem**, "If *f(x, y, z)* is **continuous on *D*,** when *D* = {(*x, y, z*) : *a* ≤ *x* ≤ *b*, *g₁*(*x*) ≤ *y* ≤ *g₂*(*x*), *h₁*(*x, y*) ≤ *z* ≤ *h₂*(*x, y*)}, then we have the equation of the form given below. In addition, all the other orders of integration too will return the equivalent end result".

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

5. The term **triple integrals** state that these are the **integrals** over a **three-dimensional region**. For computing the volumes follow the given method.

For **approximating** the **volume** in **three dimensions**, **divide** the **three-dimensional region** into **small rectangular boxes** such that each is defined as, Δ*x* × Δ*y* × Δ*z* with **volume** Δ*x* Δ*y* Δ*z*

Adding all of them and taking the **limit** the following **integral** is obtained:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz\, dy\, dx \qquad \ldots(12)$$

When the limits are constant then the volume for the rectangular box is easily computed.

Example 5. Evaluate the given triple integral of the form,

$$\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dx dy dz.$$

Solution: Follow the steps given below for evaluating the triple integral.

We have,

$$\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dx dy dz \quad = \quad 2.$$

Integrating we have,

$$= \int_1^2 \int_0^3 \left(\frac{x^2}{2} + x\left(z^2 - y^2\right)\right)\Bigg|_{x=0}^{x=2} dy dz$$

$$= \int_1^2 \int_0^3 \left(2 + 2\left(z^2 - y^2\right)\right) dy dz$$

$$= 2 \int_1^2 \int_0^3 \left(1 + z^2 - y^2\right) dy dz$$

$$= 2 \int_1^2 \left(y\left(1 + z^2\right) - \frac{y^3}{3}\Bigg|_{y=0}^{y=3}\right) dz$$

$$= 2 \int_1^2 \left(3\left(1 + z^2\right) - 9\right) dz$$

$$= 6 \int_1^2 \left(z^2 - 2\right) dz$$

$$= 2$$

Therefore,

$$\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dx dy dz \quad = \quad 2.$$

Example 6. Evaluate using the Fubini's Theorem the given triple integral of the form,

$$\int_0^1 \int_0^y \int_x^y xyz \, dz \, dx \, dy$$

Solution: Follow the steps given below for evaluating the triple integral using the Fubini's Theorem.

Given is integral,

$$\int_0^1 \int_0^y \int_x^y xyz \, dz \, dx \, dy$$

Integrating we obtain,

$$= \int_0^1 \int_0^y \left[\frac{1}{2}xyz^2 \right] \Big|_{z=x}^{y} dx \, dy$$

$$= \int_0^1 \int_0^y \left(\frac{1}{2}xy^3 - \frac{1}{2}x^3 y \right) dx \, dy$$

$$= \int_0^1 \left(\frac{1}{4}x^2 y^3 - \frac{1}{8}x^4 y \right) \Big|_{x=0}^{y} dy$$

$$= 1/48$$

Evaluation of Triple Integral

Follow the steps given below to evaluate any given triple integral.

Step 1. For evaluating the triple integral, the region of integration is determined.

Step 2. Select the order of integration for determining the limits of integration. The specified region can be divided into several subregions based on the selected order of integration.

Step 3. Precisely, when the order of integration is *dz dy dx* or *dz dx dy* then we consider the *xy*-plane, when the order of integration is *dy dz dx* or *dy dx dz* then we consider the *xz*-plane, and when the order of integration is *dx dz dy* or *dx dy dz* then we consider the *yz*-plane.

Step 4. The bounds of the outer variable has to be constant, while the bounds of the middle variable must be determined by the outer variable but the bounds of the inner variable will be determined by both the other variables.

Step 5. Finally, evaluate the each specified iterated integral in the form of single-variable integral as the appropriate variable.

Step 6. Consider all other variables of integration as constant excluding the current variable of integration.

Theorem for Evaluating the Iterated Integrals

The **Fubini's Theorem** defines the specified region which is stated as simple with regard to the order of integration *dz dy dx*.

Theorem. Let 'f' be **continuous** on a **solid region** 'Q' defined by,

$$a \leq x \leq b, \ h_1(x) \leq y \leq h_2(x), \ g_1(x, y) \leq z \leq g_2(x, y)$$

Here h_1, h_2, g_1 and g_2 are considered as the **continuous functions**. Therefore, we obtain the following equation:

$$\iiint\limits_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx.$$

…(13)

For evaluating the given triple iterated integral of the specified order of integration $dz \, dy \, dx$, both 'x' and 'y' are considered as constant for integrating the innermost integral. Then, for the second integration 'x' is considered constant.

Example 7. Evaluate the given triple iterated integral of the form,

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z) \, dz \, dy \, dx.$$

Solution: Follow the steps given below for iterating the triple iterated integral. Consider both 'x' and 'y' constant for the first integration and then integrate the given triple integral with regard to 'z'.

We have,

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z) \, dz \, dy \, dx.$$

Integrating we have,

$$= \int_0^2 \int_0^x e^x(yz + z^2) \Big]_0^{x+y} \, dy \, dx$$

$$= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2) \, dy \, dx$$

For the second integration integrate with regard to 'y' considering 'x' as constant.

Integrate the given equation second time,

$$= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2) \, dy \, dx$$

Integrating we have,

$$= \int_0^2 \left[e^x \left(x^2 y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right) \right]_0^x \, dx$$

$$= \frac{19}{6} \int_0^2 x^3 e^x \, dx$$

Now integrating with regard to '*x*' we have,

$$\frac{19}{6} \int_0^2 x^3 e^x \, dx = \frac{19}{6} \left[e^x(x^3 - 3x^2 + 6x - 6) \right]_0^2$$

$$= 19 \left(\frac{e^2}{3} + 1 \right)$$

» 65.797

8.2.3 Properties of Double and Triple Integrals

Following are the standard properties of double and triple integrals.

Properties of Double Integrals

Different properties that are defined for double integrals are also equivalent or analogous to the single integrals. These are considered as significant properties and are used for calculating the double integrals.

Property 1: Homogeneous Property

Definition. Assume that if the specified function '*f*' is integrable over a closed region '*R*' and '*k*' is an arbitrary constant then '*kf*' is integrable over the region '*R*' and we define the equation,

$$\iint_R k f(x, y) \, dx \, dy = k \iint_R f(x, y) \, dx \, dy.$$

Property 2: Additive Property

Definition. Assume that if the functions '*f*' and '*g*' are integrable over a closed region '*R*', then '*f* + *g*' is integrable over the closed region '*R*' and we define the equation,

$$\iint_R (f(x, y) + g(x, y)) \, dx \, dy = \iint_R f(x, y) \, dx \, dy + \iint_R g(x, y) \, dx \, dy.$$

Property 3: Additivity Property

Definition. Consider that '*R*' and '*S*' be the two non-overlapping closed regions (Refer Figure 8.3) and assume that a specified function '*f*' is integrable over the region '*R* ∪ *S*', then we define the equation,

$$\iint\limits_{R\cup S} f(x,y) \; \mathrm{d}x \; \mathrm{d}y = \iint\limits_{R} f(x,y) \; \mathrm{d}x \; \mathrm{d}y + \iint\limits_{S} f(x,y) \; \mathrm{d}x \; \mathrm{d}y$$

NOTES
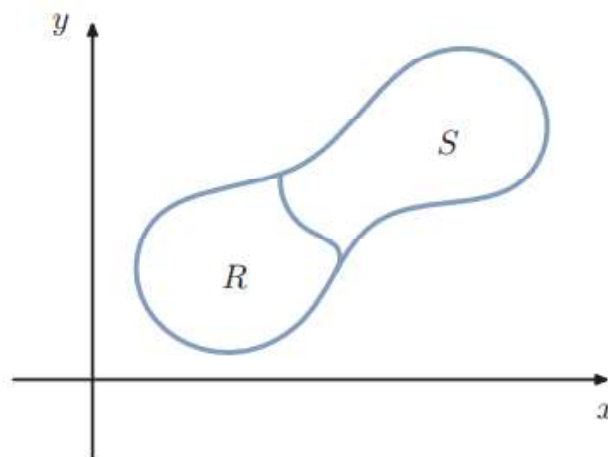
Fig. 8.3 *Two Non-Overlapping Regions 'R' and 'S'*

Property 4: When '*S*' is a Closed Subregion of '*R*' Property

Definition. Assume that the specified function '*f*' is integrable over the closed region '*R*' and also assume that '*S*' is a closed subregion of '*R*' as shown in Figure 8.4, then we define the equation,

$$\iint\limits_{S} f(x,y) \; \mathrm{d}x \; \mathrm{d}y \leq \iint\limits_{R} f(x,y) \; \mathrm{d}x \; \mathrm{d}y$$

Figure 8.4 defines the relationship between the regions '*R*' and '*S*', i.e., region '*S*' in the subregion of '*R*'.
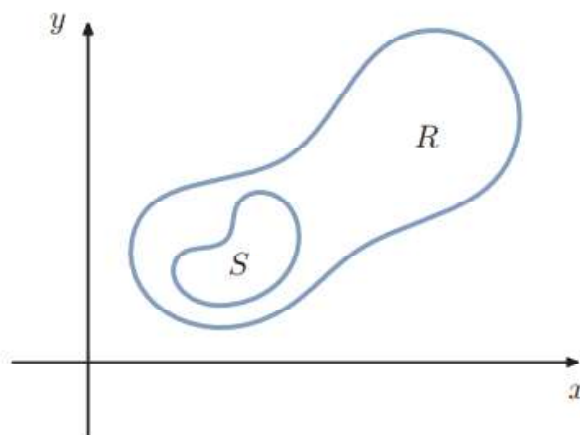
Fig. 8.4 *Relationship Between the Regions 'R' and 'S'*

Property 5: Non-Negativity of the Double Integral Property

Definition. Assume that the specified function 'f' is integrable over the closed region 'R' and let $f(x, y) \geq 0$ over 'R', then we define the equation,

$$\iint\limits_{R} f(x, y) \, dx \, dy \geq 0.$$

Property 6: Monotone Property of the Double Integral

Definition. Assume that the specified functions 'f' and 'g' are integrable over the closed region 'R' through $g(x, y) \leq f(x, y)$ for all $[x, y] \in R$, then we define the equation,

$$\iint\limits_{R} g(x, y) \, dx \, dy \leq \iint\limits_{R} f(x, y) \, dx \, dy.$$

Property 7: Median Property of the Double Integral

Definition. Assume that if 'f' is a continuous function on the closed region 'R' and $A(R)$ is the area of 'R' then there exists at least one point $[x_i, y_i] \in R$ such that we obtain the equation,

$$\iint\limits_{R} f(x, y) \, dx \, dy = f(x_i, y_j) \cdot A(R)$$

Properties of Triple Integrals

Different properties that are defined for single and double integrals are also applicable to the triple integrals. These are considered as significant properties and are used for calculating the triple integrals.

Property 1: Volume Property of the Triple Integral

Definition. The volume property is used for estimating the specified volume 'T' using the equation,

$$\iiint\limits_{T} dx \, dy \, dz = \text{Volume of } T.$$

This property is used for evaluating the triple integral specifically when 'T' is the box and defined as $T = [a, b] \times [c, d] \times [e, f]$, then,

$$\iiint_{T} dx \, dy \, dz = (b - a)(d - c)(f - e)$$

Property 2: Linearity Property of the Triple Integral

Definition. The linearity property is used for estimating the specified '*T*' using the equation where 'α' and 'β' are constants,

$$\int \int \int_T [\alpha f(x, y, z) + \beta g(x, y, z)] \, dx \, dy \, dz$$

$$= \alpha \int \int \int_T f(x, y, z) \, dx \, dy \, dz + \beta \int \int \int_T g(x, y, z) \, dx \, dy \, dz$$

Property 3: Additivity Property of the Triple Integral

Definition. The additivity property is used for evaluating the triple integral when '*T*' is divided into the finite number of non-overlapping simple and basic regions named as $T_1, \ldots, T_n$, then we define the equation,

$$\int \int \int_T f(x, y, z) \, dx \, dy \, dz$$

$$= \int \int \int_{T_1} f(x, y, z) \, dx \, dy \, dz + \ldots + \int \int \int_{T_n} f(x, y, z) \, dx \, dy \, dz.$$

Property 4: Additive Property of the Triple Integral for Evaluating Two Non-Overlapping Solid Subregions

Definition. For evaluating two non-overlapping solid subregions 'Q_1' and 'Q_2' which together comprise 'Q' we define the equation,

$$\int \int \int_Q f(x, y, z) \, dV = \int \int \int_{Q_1} f(x, y, z) \, dV + \int \int \int_{Q_2} f(x, y, z) \, dV$$

This property is used for evaluating two non-overlapping solid subregions 'Q_1' and 'Q_2' which together comprise 'Q'. When 'Q' is the simple solid region then the triple integral of the form ''$\iiint f(x, y, z) \, dV$' is evaluated with regard to the iterated integral selecting one order of integration from the possible six orders of integration stated below.

$$dx \; dy \; dz, \quad dy \; dx \; dz, \quad dz \; dx \; dy, \quad dx \; dz \; dy, \quad dy \; dz \; dx, \quad dz \; dy \; dx$$

Check Your Progress

1. What are double integrals?
2. What is the sum rule of double integrals?
3. What is the additivity property of the triple integral?
4. What is the linearity property of the triple integral?

8.3   JACOBIAN

The Jacobian transformation is used for changing variables in both the double and triple integrals.

Double Integral the Jacobian Transformation

For **changing variables** in the specified **double integral** the **Jacobian** transformation is used. The given definition is for the Jacobian transformation.

Definition. The **Jacobian** of the transformation $x = g(u, v)$ and $y = h(u, v)$ is,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \qquad \ldots (14)$$

The Jacobian is well-defined through the **determinant** of **'2 × 2' matrix**, where the determinant is computed as follows.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Consequently, the determinant formula is written as,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variable for a Double Integral

We use the Jacobian transformation method for changing the variables for a double integral.

Consider that the function $f(x, y)$ is to be integrated over the specified region 'R'. Applying the transformation $x = g(u, v)$ and $y = h(u, v)$ we can define that the region becomes 'S' and the integral takes the form,

$$\iint\limits_{R} f(x, y)\, dA = \iint\limits_{S} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| d\bar{A}$$

Here dA in the above mentioned **integral '*u/v*'** signify that it can be stated as *du* and *dv* when it is specifically converted into two single integrals instead of the form *dx* and *dy* used for *dA*. Furthermore, we normally use the *dA* notation for both.

Triple Integral the Jacobian Transformation

For **changing variables** in the specified **triple integral** the **Jacobian** transformation is used. The given definition is for the Jacobian transformation.

Definition. Given a **region '*R*'** for the **triple integral** we use the **Jacobian** transformation $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = k(u, v, w)$ for transforming the **region '*R*'** into '*S*' as **new region**. Applying Jacobian we obtain,

$$\frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

The Jacobian is well-defined through the **determinant** of **'3 × 3' matrix**, where the determinant is computed as follows.

The integral in this transformation is defined as,

$$\iiint\limits_{R} f(x, y, z)\, dV$$

$$= \iiint\limits_{S} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| d\overline{V}$$

Similar to the double integral, here *dV* used for the **integral '*u/v/w*'** signify that it can be stated for *du*, *dv* and *dw* when it is converted into three single integrals. Furthermore, we normally use the *dV* notation for both.

Example 8. Prove that when changing the variables of double integral to the polar coordinates form, we obtain $dA = r\, dr\, d\theta$.

Solution: To change the variables of double integral to the polar coordinates form, follow the steps given below.

To integrate with regard to polar coordinate, the transformation will follow the standard formula for conversion, such as,

$$x = r \cos \theta \qquad y = r \sin \theta$$

The transformation as per Jacobian is,

$$\frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

Substituting the polar coordinates we obtain,

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - \left(-r\sin^2\theta\right)$$
$$= r\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r$$

Therefore, substituting with '*r*', we obtain,

$$dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\, d\theta$$

$$= |r|\, dr\, d\theta\ r\, dr\, d\theta$$

Hence proved.

8.4 CHANGE OF ORDER OF INTEGRATION

For changing the order of integration for any given multiple integral, follow the steps given below.

Step 1. Outline the specified region of integration.

Step 2. Divide (slice) the specified region in accordance with the new defined order. Now starting with the outer variable define the new limits of integration one by one in sequence.

Step 3. Now calculate the new specified integral.

For integrating the integral $\int\int_R \boldsymbol{f}(x,y)dA$ we evaluate by defining the order of integration, i.e., first integrating with regard to '*x*' and integrating with regard to '*y*' and on changing the order of integration we first integrate with regard to '*y*' and then integrate with regard to '*x*'.

Example 9. Evaluate the given double integral by changing the order of integration by first sketching the region '*R*' for the integration of,

$$\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dx\, dy$$

Then change the order of integration for evaluating the resultant integral.

Solution: Follow the steps given below.

We first sketch or outline the region of integration as shown below in Figure 8.5.
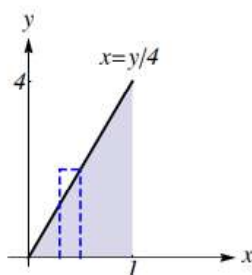
Fig. 8.5 *Region of Integration*

Figure 8.5 illustrates the region of integration as triangular region '*R*'. For changing the order of integration to *dy dx*, divide (make partition) the interval [0, 1] into comparatively small subintervals on the *x*-axis. When the order of integration is defines as *dx dy*, then on the *y*-axis divide the interval as [0, 4].

As shown in Figure 8.5, outline a rectangle with its base as the subinterval on the divided or defined partition while its length must extend or spread starting from the lower limit or edge up to the upper limit or edge of the specified region.

Subsequently, because the order of integration is defined as *dy dx*, hence every single point (x, y) on the specified rectangle essentially satisfy,

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 4x$$

Given is the integral,

$$\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy$$

On integrating we obtain,

$$\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy \quad = \quad \int_0^1 \int_0^{4x} \sin(\pi x^2)dydx$$

$$= \quad \int_0^1 y\sin(\pi x^2)\Big|_{y=0}^{y=4x} dx$$

$$= \quad \int_0^1 4x\sin(\pi x^2)dx$$

$$= \quad -\frac{2}{\pi}\cos(\pi x^2)\Big|_{x=0}^{x=1}$$

$$\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy \quad = \quad \frac{4}{\pi}$$

Example 10. Reverse the order of integration for the given double integral of the form,

$$\int_0^1 \int_{x^2}^x xy\, dx\, dy$$

Solution: Follow the steps given below.

Step 1. The specified region is defined through $0 \le x \le 1$ and $x^2 \le y \le x$. The present order of integration is shown in Figure 8.6 having the horizontal portions or slices.
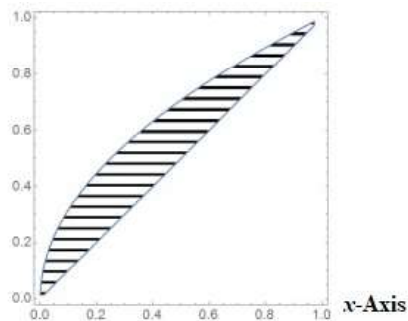
Fig. 8.6 Order of Integration having the Horizontal Portions or Slices

Step 2. Now reversing the order of integration, the specified region is partitioned or sliced such that the partitioned slices have vertical portions and are perpendicular to the *x*-axis as shown in the Figure 8.7.
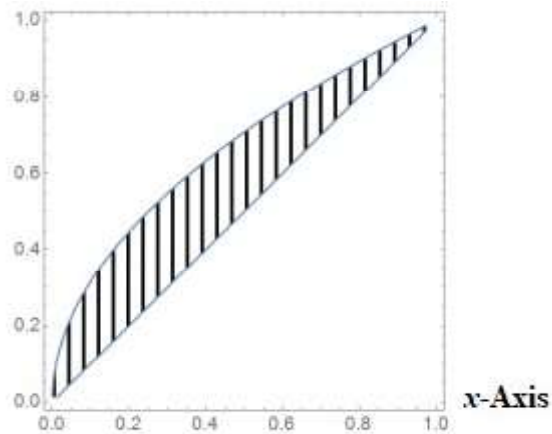
Fig. 8.7 Reverse Order of Integration having the Vertical Portions or Slices

Step 3. The specified range for '*x*' is $0 \le x \le 1$, and consequently the new limits of '*y*' will be $x \le y \le \sqrt{x}$.

Step 4. Therefore, we obtain the integral with reverse order of integration as follows,

$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$$

Check Your Progress

5. Which transformation is used for changing variables in both the double and triple integrals?

6. Define Jacobian transformation.

8.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Double integrals are the integrals using which we can find the function of two variables.

2. $\iint_D f(x, y) + g(x, y)dA = \iint_D f(x, y)dA + \iint_D g(x, y)dA$

3. $\int\int\int_T f(x, y, z)\, dx\, dy\, dz$

$$= \int\int\int_{T_1} f(x, y, z)\, dx\, dy\, dz + \ldots + \int\int\int_{T_n} f(x, y, z)\, dx\, dy\, dz.$$

4. $\int\int\int_T [\alpha f(x, y, z) + \beta g(x, y, z)]\, dx\, dy\, dz$

$$= \alpha \int\int\int_T f(x, y, z)\, dx\, dy\, dz + \beta \int\int\int_T g(x, y, z)\, dx\, dy\, dz$$

5. The Jacobian transformation is used for changing variables in both the double and triple integrals.

6. The Jacobian of the transformation $x = g(u, v)$ and $y = h(u, v)$ is,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

8.6 SUMMARY

- The multiple integral is defined as a 'definite integral' of a function of more than one real variable, such as $f(x, y)$ or $f(x, y, z)$. Integrals of a function of 'two variables' over a region in $R2$ are termed as 'double integrals' while the integrals of a function of 'three variables' over a region in $R3$ are termed as 'triple integrals'.

- A double integral has the form,

$$\iint_R f(x, y)\, dx\, dy$$

  Here 'R' is termed as the region of integration, precisely the region in the (x, y) plane. Fundamentally, the double integral evaluates the volume under the specified surface $z = f(x, y)$.

- Integration of a function of three variables, say $w = f(x, y, z)$ over a three-dimensional region R in xyz-space is termed a triple integral and is denoted as follows,

$$\iiint_R f(x, y, z)\, dV$$

- The Jacobian of the transformation $x = g(u, v)$ and $y = h(u, v)$ is,

$$\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

8.7 KEY WORDS

- **Domain:** The domain of a function is the complete set of possible values of the independent variable.

- **Continuous functions:** A continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.

- **Partial derivative:** a derivative of a function of two or more variables with respect to one variable, the other(s) being treated as constant.

8.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Evaluate the given double integral of the form,

$$\int_{y=-1}^{1} \int_{x=1}^{3} (x - 2y)\, dx\, dy$$

2. Evaluate the given double integral of the form, $\int_{0}^{1} \int_{1}^{3} xy \sin x\, dx\, dy$.

3. Define additivity property of the triple integral.

4. Define monotone property of the double integral.

Long Answer Questions

1. Evaluate the given triple iterated integral of the form,

$$\int_{0}^{1} \int_{0}^{-1} \int_{1}^{x} (x + y + z)\, dx\, dy\, dz.$$

2. Let $x = r \cos \theta$, $y = r \sin \theta$ be the change of coordinates from (r, θ) to (x, y). What is the Jacobian of this transformation?

3. Evaluate the given double integral by changing the order of integration by first sketching the region 'R' for the integration of,

$$\int_{0}^{1} \int_{-1}^{1} x \sin y\, dx\, dy.$$

4. Reverse the order of integration for the given double integral of the form,

$$\int_{-1}^{1} \int_{2}^{3} xy\, dx\, dy.$$

8.9 FURTHER READINGS

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BLOCK - III
BETA, GAMMA FUNCTIONS AND SOLUTIONS
OF DIFFERENTIAL EQUATIONS

UNIT 9   BETA AND GAMMA FUNCTIONS

Structure

9.0   INTRODUCTION

In this unit, you will learn the definition of gamma function and the beta function. Further, you will know about some of the properties of gamma and beta function. The gamma function's original intent was to model and interpolate the factorial function, mathematicians and geometers have discovered and developed many other interesting applications. The gamma function is a component in various probability-distribution functions, and as such it is applicable in the fields of probability and statistics, as well as combinatorics. The Beta function was first studied by Euler and Legendre and was given its name by Jacques Binet; its symbol *B* is a Greek capital â rather than the similar Latin capital β. Just as the gamma function for integers describes factorials, the beta function can define a binomial coefficient after adjusting indices.

9.1   OBJECTIVES

After going through this unit, you will be able to:

- Define gamma function
- Discuss properties of gamma function
- Define beta function
- Discuss properties of beta function

9.2   GAMMA FUNCTION

Mathematically, the **Gamma function $\Gamma(z)$** or also sometimes as $\Gamma(x)$, is defined as, 'the **extension** of the **factorial function**, through its argument which is shifted down by 1 for the real and complex numbers', and is represented using the Greek alphabet Γ. When **'n'** is a positive integer then,

$$\Gamma(n) = (n-1)!, \text{ for } n = 0, 1, 2, \ldots \ldots$$

The Gamma function can be specified for all complex numbers excluding the non-positive integers. Thus, we can define the complex numbers by means of a positive real part through the convergent improper integral as follows,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

Definition 1. The **Gamma function** can be defined as the generalization of **$n!$** (n factorial), where **'n'** is any positive integer to **'$x!$'** and **'x'** being any real number. The Gamma function defined by the **improper integral** is,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad\qquad \ldots(1)$$

Following are the fundamental properties of Gamma function.

- The **Gamma function** is **convergent** for $x > 0$.
- The following important property for Gamma function is obtained by using the method **integration by parts** to Equation (1).

$$\Gamma(x+1) \;=\; x\,\Gamma(x) \qquad\qquad \ldots(2)$$

The **function $\Gamma(x)$** in Equation (2) is calculated for all $x > 0$ when we know its value for the **interval $1 \leq x < 2$**.

Definition 2. The following **improper integral** exists for every $x > 0$, such that,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

Here the **function $\Gamma(x)$** is termed as the **Gamma function**.

We will define improper integrals with regard to the following theorems.

Theorem 1. Comparison Test for Improper Integral of Type I

Let $f(x)$ and $g(x)$ are two continuous functions on **$[a, \infty]$** such that for all $x \geq a$,

$$0 \;\leq\; f(x) \;\leq g(x)$$

Then, we can state,

1. When $\displaystyle\int_a^\infty g(x)\,dx$ is **convergent** accordingly is $\displaystyle\int_a^\infty f(x)\,dx$.

2. When $\displaystyle\int_a^\infty f(x)dx$ is **divergent to infinity** accordingly is $\displaystyle\int_a^\infty g(x)dx$.

Theorem 2. Limit Comparison Test

Let $f(x)$ and $g(x)$ are two non-negative continuous functions on $[a, \infty]$. Assume that,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \text{ with } L > 0.$$

Consequently, both $\displaystyle\int_a^\infty g(x)dx$ and $\displaystyle\int_a^\infty f(x)dx$ are either **convergent** or **divergent**.

Theorem 3. For every $x > 0$, the following improper integral is considered as convergent.

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

Proposition 1. For $x > 0$,

$$\Gamma(x + 1) = x\,\Gamma(x)$$

Proof. For every $N > 0$, through integration by parts, we can define,

$$\int_0^N e^{-t}t^x dt = -t^x e^{-t}\Big|_0^N + x \int_0^N e^{-t}t^{x-1}dt$$

$$= -N^x e^{-N} + x \int_0^N e^{-t}t^{x-1}dt.$$

For all $x > 0$, we can define that limit is,

$$\lim_{N \to \infty} N^x e^{-N} = \lim_{N \to \infty} \frac{N^x}{e^N} = 0$$

Accordingly, integration by parts gives:

$$\Gamma(x + 1) = \lim_{N \to \infty} \int_0^N e^{-t}t^x dt$$

$$= \lim_{N \to \infty} \left(-N^x e^{-N} + x \int_0^N e^{-t}t^{x-1}dt\right)$$

$$= x \lim_{N \to \infty} \int_0^N e^{-t}t^{x-1}dt$$

$$= x\,\Gamma(x)$$

Hence proved.

Evidently,

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

As proved by **Proposition 1**, for $x > 0$, through **integration by parts** we obtain,

$$\Gamma(x + 1) = x\,\Gamma(x)$$

Here $\Gamma(x + 1) = x\,\Gamma(x)$ is termed as significant **functional equation** and the **integer values** of the functional equation is,

$$\Gamma(n + 1) = n\,!$$

Corollary 1. For every $n \geq 0$, $\Gamma(n + 1) = n!$ for $n = 0, 1, 2, \ldots\ldots$.

The **factorial $n!$** of positive whole number n then this integer n is represented by,

$$n! = 1 \times 2 \times 3 \times \ldots\ldots \times (n - 1) \times n$$

For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

However this formula is not valid if n is not an integer.

Indefinite, Proper Definite and Improper Definite Integrals

The *integration process* includes mainly two types of integrals, the *definite integrals* and the *indefinite integrals*. Between these two integrals the fundamental difference is that the *definite integrals* are defined with regard to *limits of integration* while the *indefinite integrals* are *not* defined with regard to *limits*. Further, the definite integrals are classified as the *proper definite integrals* and the *improper definite integrals*. The definite integrals are evaluated by simply using real numbers as limits of integration. In the integration process you can *set* the *upper limits* of *integration* to '∞' or infinity and the *lower limits* can be *set* to '$-\infty$' or negative infinity. Following table describes the types of an indefinite integral, a proper definite integral and an improper definite integral.

Indefinite Integral	Proper Definite Integral	Improper Definite Integral
$\int e^{-t}\,dt$	$\int_0^1 e^{-t}\,dt$	$\int_0^\infty e^{-t}\,dt$

The *definite integral* which contains one or more additional *infinite limits* of *integration* or an *integrand* which contains *infinity* (∞) within its *limits* of *integration* is termed as an *improper integral*.

Definition 3. According to **Euler (1730)**, when $x > 0$ then,

$$\Gamma(x) = \int_0^1 (-\log(t))^{x-1}\,dt. \qquad \ldots(3)$$

Euler stated that for **basic change of variables** this definition or in **Equation (3)** for $x > 0$ will take the fundamental form of Gamma equation.

Theorem 4. For $x > 0$, we have,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$ …(4)

Also sometimes,

$$\Gamma(x) = 2 \int_0^\infty t^{2x-1} e^{-t^2} dt.$$ …(5)

Theorem 4 for **Gamma function $\Gamma(x)$** is also termed as the **Eulerian Integral of the Second Kind**. We can deduce the following derivatives by integrating under the integral sign of Equation (4).

$$\Gamma'(x) = \int_0^\infty t^{x-1} e^{-t} \log(t) dt.$$

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1} e^{-t} \log^n(t) dt$$

Evaluating the Integral

For **positive 'x'** the Gamma function is defined through the integral,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

To evaluate this integral we first consider as $x = 1$.

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} dt$$

$$= \int_0^\infty e^{-t} dt$$

$$= \left[-e^{-t} \right]_0^\infty$$

$$= \lim_{t \to \infty} \left(\frac{-1}{e^t} \right) - (-1)$$

$$= 0 + 1 = 1$$

Now considering that $x = 1/2$ we have,

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt$$

$$= \begin{vmatrix} y = \sqrt{t} \\ 2dy = t^{-1/2} \, dt \\ t = y^2 \end{vmatrix}$$

$$= 2 \int_0^\infty e^{-y^2} \, dy = \sqrt{\pi}.$$

To evaluate the value for the Gamma function at some additional points we use the identity integration by parts as shown below.

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} \, dt$$

$$= \begin{vmatrix} f = t^x & g' = e^{-t} \\ f' = x\,t^{x-1} & g = -e^{-t} \end{vmatrix}$$

$$= \left[-t^x e^{-t} \right]_0^\infty - \int_0^\infty -x\,t^{x-1} e^{-t} \, dt$$

$$= -\lim_{t \to \infty} \left(\frac{t^x}{e^t} \right) + x \int_0^\infty t^{x-1} e^{-t} \, dt$$

$$= 0 + x\,\Gamma(x) = x\,\Gamma(x)$$

Therefore, for **positive x** we can define,

$$\Gamma(x + 1) = x\,\Gamma(x)$$

Applying this to the **positive integer 'n'** we obtain,

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1) \cdot (n-2)\Gamma(n-2) = \ldots$$
$$= (n-1) \cdot (n-2) \cdots 2 \cdot 1 \cdot \Gamma(1) = (n-1)!.$$

Therefore, we can state that, '*Gamma function is the generalization of the factorial function*'. Figure 9.1 illustrates the limit of the Gamma function at **'0'** on the right-hand side of the graph is **'∞'** or infinity.
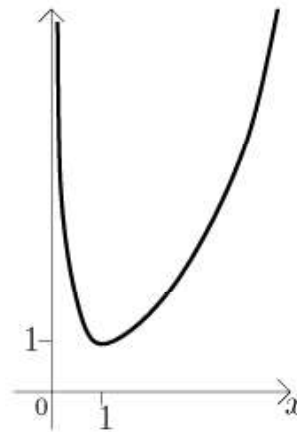
Fig. 9.1 *Limit of the Gamma Function*

Unique Standards of $\Gamma(x)$

Excluding the integer values for $x = n$, which has,

$\qquad \Gamma(n) = (n-1)!$

While certain non-integer values are defined as closed form.

With **change of variables** for '$t = u^2$', we obtain the given equation,

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt$$

$$= 2 \int_0^\infty e^{-u^2} du$$

$$= 2 \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

The various forms of functional equation for positive integers 'n' involves:

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1.3.5...(2n-1)}{2^n} \sqrt{\pi}$$

$$\Gamma\left(n + \frac{1}{3}\right) = \frac{1.4.7...(3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(n + \frac{1}{4}\right) = \frac{1.5.9...(4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right)$$

Similarly, for the negative integers,

$$\Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n 2^n}{1.3.5...(2n-1)} \sqrt{\pi}.$$

Another explanation by **Euler (1729)** and **Gauss (1811)** is as follows.

Assume that $x > 0$ and state,

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)...(x+p)}$$

$$= \frac{p^x}{x(1 + x/1)...(1 + x/p)}$$

Then,

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x).$$

Evidently,

$$\Gamma_p(1) = \frac{p!}{1(1 + 1)...(1 + p)}p = \frac{p}{p + 1}$$

And also,

$$\Gamma_p(x + 1) = \frac{p!p^{x+1}}{(x + 1)...(x + p + 1)}$$

$$= \frac{p}{x + p + 1}x\Gamma_p(x)$$

Therefore,

$$\Gamma(1) = 1$$

$$\Gamma(x + 1) = x\Gamma(x)$$

9.2.1 Properties of Gamma Function

The **Gamma function $\Gamma(z)$** follows the below given properties:

Gamma Difference Equation

$$\Gamma(z + 1) = z(z)$$

Euler's Gamma Function

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt \quad (z \in \mathbb{C}, \operatorname{Re} z > 0).$$

Euler's Reflection Formula

$$\forall z \notin \mathbb{Z} : \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$$

Legendre's Duplication Formula

$$\forall z \notin \left\{-\frac{n}{2} : n \in \mathbb{N}\right\} : \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

Here '$\mathbb{N}$' specifies the natural numbers.

Also represented as,

$$\Gamma(x)\Gamma(x + 1/2) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x).$$

Complex Conjugate

$$\forall z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} : \Gamma(\bar{z}) = \overline{\Gamma(z)}$$

Gamma Function of the One Half

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Complement Formula

The complement formula is significant identity concerning the Gamma function through the complementary values '**x**' and '**1 − x**' which must not be negative or null integers represented as,

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}.$$

Gauss Multiplication Formula

$$\Gamma(x)\Gamma\left(x + \frac{1}{n}\right)\Gamma\left(x + \frac{2}{n}\right)\ldots\Gamma\left(x + \frac{n-1}{n}\right)$$

$$= (2\pi)^{(n-1)/2} n^{1/2-nx}\Gamma(nx)$$

Euler Multiplication Formula

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

Stirling Formula

When the **integer '*n*'** tends to **infinite** then we use the following **asymptotic formula**,

$$\Gamma(n + 1) = n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

The Stirling formula is significant since the arithmetical factorial function is considered equivalent to that specific expression which comprises of important and essential analytic constants, such as $\sqrt{2}, \pi, e$.

Recursive Formula

The Gamma function satisfies the recursive property and is represented as.

$$\Gamma(z) = (z-1) \ \Gamma(z-1)$$

Or,

$$\Gamma(\alpha) = (\alpha-1) \ \Gamma(\alpha-1)$$

Use the method integration by parts for deriving the recursion as shown below.

$$\Gamma(z) = \int_0^\infty x^{z-1} \exp(-x)dx$$

$$= \left[-x^{z-1} \exp(-x) \right]_0^\infty + \int_0^\infty (z-1)x^{z-2} \exp(-x)dx$$

$$= (0-0) + (z-1) \int_0^\infty x^{(z-1)-1} \exp(-x)dx$$

$$= (z-1)\Gamma(z-1)$$

Example 1. Evaluate the values for $x = 1/2, x = 1/3, x = 1/4$ applying the formula,

$$\Gamma(x)\Gamma(1-x)$$

Solution: The values are evaluated as follows:

1.   For $x = 1/2$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2.   For $x = 1/3$

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi\sqrt{3}}{3}$$

3.   For $x = 1/4$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}.$$

Integers and Half-Integers for Gamma Function

When estimating the positive integers the Gamma function agrees with the factorial fractions.

The gamma function, $\Gamma(n) = (n-1)!$, where $n = 1, 2, 3, 4, \ldots$, and so on.

Therefore, the factorial is estimated as,

$$\Gamma(n) = (n-1)!$$
$$\Gamma(1) = 1$$
$$\Gamma(2) = 1$$

$\Gamma\ (3)\ =\ 2$

$\Gamma\ (4)\ =\ 6$

$\Gamma\ (5)\ =\ 24,\ \ldots\ldots,$ and so on.

The Gamma function is not distinct for non-positive integers. For half-integers which are positive, the values of functions are accurately given as,

$$\Gamma\left(\tfrac{n}{2}\right) = \sqrt{\pi}\frac{(n-2)!!}{2^{\frac{n-1}{2}}}$$

Equivalently, we can define the values of non-negative integer 'n' as follows:

$$\Gamma\left(\tfrac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{1}{2}-n\right) = \frac{(-2)^n}{(2n-1)!!}\sqrt{\pi} = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}$$

Here '**n!!**' represents the double factorial.

Example 2. Evaluate the values for $\Gamma(5)$.

Solution: The values are evaluated as follows using the Gamma factorial function,

$\Gamma\ (n)\ =\ (n-1)!$

Taking $n = 5$, we have,

$\Gamma\ (5)\ =\ (5-1)!$

$= 4!$

$= 4 \times 3 \times 2 \times 1$

$= 24$

Example 3. Evaluate the values for the given ratio.

$$\frac{\Gamma\left(\frac{16}{3}\right)}{\Gamma\left(\frac{10}{3}\right)}$$

Solution: The values are evaluated as follows using the recursive formula:

$$\Gamma\ (z)\ =(z-1)\ \Gamma\ (z-1)$$

Applying the recursive formula to the numerator of the given ratio, we obtain,

$$\frac{\Gamma\left(\frac{16}{3}\right)}{\Gamma\left(\frac{10}{3}\right)} = \frac{\left(\frac{16}{3}-1\right)\Gamma\left(\frac{16}{3}-1\right)}{\Gamma\left(\frac{10}{3}\right)}$$

$$= \frac{13}{3} \frac{\Gamma\left(\frac{13}{3}\right)}{\Gamma\left(\frac{10}{3}\right)}$$

$$= \frac{13}{3} \frac{\left(\frac{13}{3} - 1\right)\Gamma\left(\frac{13}{3} - 1\right)}{\Gamma\left(\frac{10}{3}\right)}$$

$$= \frac{13}{3} \frac{10}{3} \frac{\Gamma\left(\frac{10}{3}\right)}{\Gamma\left(\frac{10}{3}\right)}$$

$$= 130/9$$

Check Your Progress

1. For what value of x, the gamma function is convergent?
2. Gamma function is the generalization of which function?
3. What is Gamma difference equation?

9.3 BETA FUNCTION

In mathematics, the **beta function** is defined as a special characteristic function and is also characterized (categorized) as the **'Euler integral of first kind'**. This function was first introduced by Euler and Legendre.

The mathematician Jacques Binet termed it as **'beta'** and defined the beta function using the **symbol 'B'** taken from Greek language capitals B or uppercase beta. Following are the standard definitions of beta function.

Definition 1. The **beta function** is defined on the fields of real numbers and is usually denoted by $B(p, q)$, where p and q are **real numbers** and is represented as,

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\, dt$$

For $p, q > 0$.

We can also represent the beta function in the Gamma function form as follows.

$$B(p, q) = \frac{\Gamma p.\Gamma q}{\Gamma(p+q)}$$

Since the Gamma function is represented as,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

The factorial notation form can be used to calculate the beta function using the formula,

$$B(p, q) = \frac{(p-1)!(q-1)!}{(p+q-1)!}$$

For $p! = p \cdot (p-1) \cdot (p-2) \cdot (p-3) \ldots\ldots$ and so on.

Definition 2. The **beta function** is expressed as,

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$$

When $x \geq 1$ and $y \geq 1$, then this **integral** is termed as a **proper integral**. For $x > 0$ and $y > 0$ and when **either** or **both** $x < 1$ or $y < 1$ then the **integral** is termed as **improper but convergent**.

Definition 3. The **beta function** is a **two parameter composition** of **Gamma functions** and is represented as,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

The beta function and the Gamma function can be used for representing various integrals. The following two are significant equations representing integrals.

$$\int_0^{\pi/2} \sin^{2x-1}\theta \, \cos^{2y-1}\theta \, d\theta = \frac{1}{2}B(x, y) = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\int_0^\infty \frac{x^{p-1}}{1+x} \, dx = \Gamma(p)\,\Gamma(p-1) = \frac{\pi}{\sin \pi P} \quad 0 < p < 1$$

Definition 4. The **definite integral** which is related to the Γ–**function** is the **Beta function** represented as $B(a, b)$ and is defined as,

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt.$$

For $a > 0$, $b > 0$. The **beta function** can be represented as follows in the **Gamma function format**,

$$B(a, b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a + b)}$$

Definition 5. When x, y are **positive real numbers** then the **beta function $B(x, y)$** can be represented as follows,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

Also, $B(x, y) = B(y, x)$ for $\forall\, x, y > 0$.

Definition 6. Beta function is **'Symmetric'**, i.e., the **value** of **beta function** is **irrespective** to the **order** of its **parameters**.

$$B(a, b) = B(b, a)$$

Properties of Beta Function

Following are the significant properties of the **beta function** for p and q real numbers.

1. $B(p, q) = B(q, p)$

2. $B(p, q) = B(p, q + 1) + B(p + 1, q)$

3. $B(p, q + 1) = B(p, q) \cdot \frac{q}{p+q}$

4. $B(p + 1, q) = B(p, q) \cdot \frac{p}{p+q}$

5. $B(p, q) \cdot B(p + q, 1 - q) = \frac{\pi}{p\,sin(\pi q)}$

6. $B(p, q) = \frac{\Gamma p.\Gamma q}{\Gamma(p+q)}$

7. The following are two significant integral forms of the beta function.

- $B(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\ dt$

- $B(p, q) = 2 \int_0^{\frac{\pi}{2}} sin^{2p-1}\theta\ cos^{2q-1}\ d\theta$

Example 4. Evaluate the values for $B(3/2, 1)$.

Solution: The values are evaluated as follows using the given beta function:

$$B(p, q) = \frac{(p-1)!.(q-1)!}{(p+q-1)!}$$

Given is, $p = 3/2$ and $q = 1$.

Applying the values for $p = 3/2$ and $q = 1$ in the above equation, we obtain,

$$B(\tfrac{3}{2}, 1) = \frac{(\tfrac{3}{2}-1)!.(1-1)!}{(\tfrac{3}{2}+1-1)!}$$

$$= \frac{(\tfrac{1}{2})!.(0)!}{(\tfrac{3}{2})!}$$

Evidently,

$$\left(\tfrac{1}{2}\right)! = \tfrac{1}{2}.\sqrt{\pi}$$

And,

$$\left(\tfrac{3}{2}\right)! = \tfrac{3}{2} \cdot \tfrac{1}{2}.\sqrt{\pi}$$

Hence,

$$\frac{(\tfrac{1}{2})!.(0)!}{(\tfrac{3}{2})!} = \frac{\tfrac{1}{2}.\sqrt{\pi}.1}{\tfrac{3}{2} \cdot \tfrac{1}{2}.\sqrt{\pi}}$$

$$= \frac{1}{\tfrac{3}{2}}$$

$$= \frac{2}{3}$$

Example 5. Evaluate the values for,

$$\int_0^1 t^4(1-t)^3 dt$$

Solution: The values are evaluated as follows:

Given is,

$$\int_0^1 t^4(1-t)^3 dt$$

Can be defined as,

$$\int_0^1 t^{5-1}(1-t)^{4-1}\,dt$$

We can compare this form with the beta function as follows,

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

We obtain, $p = 5$ and $q = 4$.

Consequently, using the beta function of the form,

$$B(p, q) = \frac{(p-1)!.(q-1)!}{(p+q-1)!}$$

We get,

$$\int_0^1 t^{5-1}(1-t)^{4-1}\,dt = \frac{4!.3!}{8!}$$

$$= \frac{4!.6}{8.7.6.5.4!}$$

$$= 1\,/\,8.7.5$$

$$= 1\,/\,280$$

Check Your Progress

4. Which function is known as Euler integral of first kind?

5. How will you represent beta function $B(x, y)$ when x and y are positive real numbers?

6. When does the beta function termed as improper but convergent?

9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. For $x > 0$.

2. Gamma function is the generalization of the factorial function.

3. $\Gamma(z + 1) = z\,\Gamma(z)$.

4. Beta function.

5. When x, y are positive real numbers then the beta function $B(x, y)$ can be represented as follows,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

6. The beta function is expressed as,

For $x > 0$ and $y > 0$ and when either or both $x < 1$ or $y < 1$ then the integral is termed as improper but convergent.

9.5 SUMMARY

- The Gamma function can be defined as the generalization of $n!$ (n factorial), where 'n' is any positive integer to '$x!$' and 'x' being any real number. The Gamma function defined by the improper integral is,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

- $\Gamma(x + 1) = x\, \Gamma(x)$ for x > 0.

Indefinite Integral	Proper Definite Integral	Improper Definite Integral
$\int e^{-t}\, dt$	$\int_0^1 e^{-t}\, dt$	$\int_0^\infty e^{-t}\, dt$

- $$\Gamma(x) = \int_0^1 \left(-\log(t)\right)^{x-1} dt.$$

- The beta function is defined on the fields of real numbers and is usually denoted by $B(p, q)$, where p and q are real numbers and is represented as,

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt$$

- The factorial notation form can be used to calculate the beta function using the formula, $B(p, q) = \dfrac{(p-1)!(q-1)!}{(p+q-1)!}$ for $p! = p.\,(p-1).\,(p-2).$ $(p-3)$ ……. and so on.

- The beta function is expressed as,

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

When $x \geq 1$ and $y \geq 1$, then this integral is termed as a proper integral. For

$x > 0$ and $y > 0$ and when either or both $x < 1$ or $y < 1$ then the integral is termed as improper but convergent.

- The beta function is a two parameter composition of Gamma functions and is represented as,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

- Beta function is 'Symmetric', i.e., the value of beta function is irrespective to the order of its parameters. $B(a, b) = B(b, a)$

9.6 KEY WORDS

- **Factorial:** The factorial of a non-negative integer n, denoted by n!, is the product of all positive integers less than or equal to n. For example, the value of 0! is 1.

- **Improper integral:** An improper integral is the limit of a definite integral as an endpoint of the interval of integration approaches either a specified real number, or in some instances as both endpoints approach limits.

- **Convergent:** Convergence, in mathematics, property (exhibited by certain infinite series and functions) of approaching a limit more and more closely as a variable of the function increases or decreases or as the number of terms of the series increases.

- **Divergent:** A divergent series is an infinite series that is not convergent, meaning that the infinite sequence of the partial sums of the series does not have a finite limit. If a series converges, the individual terms of the series must approach zero.

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is Gauss multiplication Formula?
2. Show that Gamma function satisfies the recursive property.
3. Write a short note on gamma function.
4. Write a short note on beta function.
5. Show that beta function is symmetric.

Long Answer Questions

1. Evaluate the values for x = 2, x = 3 applying the formula,

$$\Gamma(x)\Gamma(1-x)$$

2. Evaluate the values for $\Gamma(10)$

3. Evaluate the values for $B(5/2, 3)$.

4. Using beta function evaluate the values for, $\int_1^1 (t-9)^3 dt.$

5. Using beta function evaluate the values for, $\int_0^1 (t^2-t)^2 dt.$

6. Discuss about Gamma function and Beta function with the help of examples.

9.8 FURTHER READINGS

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UNIT 10   DIFFERENTIAL EQUATIONS

Structure

10.0 INTRODUCTION

This unit discusses about differential equations and their solutions. A differential equation relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Differential equations play a very important role in many disciplines including engineering, physics, economics, and biology.

Differential equations are called partial differential equations (pde) or ordinary differential equations (ode) according to whether or not they contain partial derivatives. The order of a differential equation is the highest order derivative occurring. A solution (or particular solution) of a differential equation of order n consists of a function defined and n times differentiable on a domain having the property that the functional equation obtained by substituting the function and its n derivatives into the differential equation holds for every point in that domain.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe partial and ordinary differential equations

- Solve differential equations

- Solve differential equations using the separation of variables method

10.2 DIFFERENTIAL EQUATIONS

In mathematics, a **Differential Equation (DE)** is defined as an equation of the form that interconnects certain function with its derivatives, where usually the function represents the physical quantity while the derivatives denote their rates of change and the relationship between the two is defined by the equation. Fundamentally, the '*solutions of differential equations*' are **functions** which precisely "*represent the relationship or correlation between a continuously varying or fluctuating quantity and its rate of change*".

Definition. A **Differential Equation (DE)** comprises of one or more expressions including **derivatives** of **one dependent variable '*y*'** with reference to another **independent variable '*x*'**, such as

$$\frac{dy}{dx} = 2x$$

Definition. An ordinary differential equation is a differential equation that includes a function of a single variable and some of its derivatives, such as

$$\frac{dy}{dx} = 2x^2 + 3x + 5$$

Definition. A differential equation is an equation for a function that relates the values of the function to the values of its derivatives.

Definition. A differential equation is an equation between specified derivative on an unknown function, its values and known quantities and functions.

Definition. A **Differential Equation (DE)** is an **equation** that comprises of a **function** and **its derivatives**. Differential equations are categorized as **Partial Differential Equations (PDE)** or **Ordinary Differential Equations (ODE)** in accordance with whether or not they hold **partial derivatives**, while the **order** of a **differential equation** is defined on the basis of the **highest order derivative** that occurs in the equation.

Several laws of physics are formulated or expressed as Differential Equations (DEs). Therefore, the greatest mathematicians and mathematical physicists have been studying DEs ever since the time of Sir Newton.

10.2.1 Types Of Differential Equations

The differential equations are of the following types.

Ordinary Differential Equation (ODE): An ordinary differential equation is specifically a differential equation that is defined for a single function. An ordinary differential equation comprises of ordinary derivatives. Therefore the ordinary differential equations manage the functions of a single variable and their derivatives.

Definition. An **Ordinary Differential Equation (ODE)** is an **equation** that contains an **unknown function** of **one real** or **complex variable 'x'**, its **derivatives** and some **specified functions** of **'x'**. Generally, the unknown function is characterized by a **variable** denoted by **'y'**, which consequently **depends on 'x'**. Hence, **'x'** is termed as the **independent variable** of the equation.

Partial Differential Equation (PDE): A partial differential equation is defined specifically for a differential equation that is a function of several variables. A partial differential equation comprises of partial derivatives.

Definition. A **Partial Differential Equation (PDE)** is a **differential equation** of the form that contains **unknown multivariable functions** and their **partial derivatives**. Therefore the partial differential equations deal with functions of several variables.

10.2.2 Order and Degree of Differential Equations

Basically, the **order** of a **differential equation** is specified as the **order** of the **highest derivative** appearing in the **equation**.

Definition Order. **Differential equations** are **categorized** with respect to **order**, since the **order** of a **differential equation** is defined with regard to the **order** of the **highest order derivative** that exists or occurs in the **equation**.

Definition Order. The **order** of the **differential equation** is defined with regard to the **order** of the **highest derivative** included in the **equation**.

The **degree** of any **differential equation** is defined with regard to the **power** of the **highest order derivative** that exists in the **equation**.

Definition Degree. The **degree** of **differential equation** of which the **differential coefficients** are **free** from **radicals** and **fractions** is termed as the **positive integral index** of the **highest power** of the **highest order derivatives** occurring in the **equation**.

The following differential equations represents the equations with different orders and degrees.

Following are examples of **first order differential equations**:

$$\frac{dy}{dx} = 2x \qquad \text{and} \quad \frac{dy}{dx} = 2x^2 + 3x + 5$$

Following are examples of **second order (order 2) differential equations**:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \qquad \text{and} \quad ay'' + by' + cy = 0$$

Following is the example of **third order (order 3) differential equation** and **degree 1**:

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\frac{d^2y}{dx^2} + e^y$$

Following is the example of **second order (order 2) differential equation** and **degree 3**:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = \sin x$$

An **ordinary differential equation** of **order 'n'** is an **equation** of the form,

$$F(x, y, y', y'', \ldots\ldots\ldots, y^{(n)}) = 0$$

Here **'y'** is a **function** of x and $y' = dy/dx$ is the **first derivative** with respect to **'x'** and similarly $y^{(n)} = d^n y/dx^n$ is the **'nth' derivative** with respect to **'x'**.

An **ordinary differential equation** of **order 'n'** is assumed to be **linear** if it is of the form,

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = Q(x)$$

A **linear ordinary differential equation** is considered as **homogeneous** when **$Q(x) = 0$**. Occasionally, the following form of an ordinary differential equation is also considered as **homogeneous**.

$$y' = f\left(\frac{y}{x}\right)$$

As a general rule, the **ordinary differential equations** of **nth-order** has **'n'** **linearly independent solutions**.

Linear Differential Equation

A **linear differential equation** is the specific form of a differential equation which is defined by means of a **linear polynomial** for the **unknown function** and **its derivatives**. Following is the form of linear differential equation:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} + b(x) = 0$$

Here, the forms $a_0(x)$, $a_1(x)$, $\ldots$, $a_n(x)$ and $b(x)$ are termed as the **arbitrary differentiable functions** which may **not be linear**, and $y', y'', \ldots, y^{(n)}$ are defined as the **successive derivatives** of the **unknown function 'y'** with regard to the **variable 'x'**.

Assume that **'y'** is a **dependent variable** while **'x'** is an **independent variable** and also that $y' = f(x)'$ is an **unknown function** of **'x'**. There are **different notations** for writing the differentiation equations which depend on the specific author, for example as per the **Leibniz** the **notation** for **differentiation** and **integration** is represented as $dy/dx, d^2y/dx^2, \ldots, d^n y/dx^n,$ while the **derivatives** are represented using the **Lagrange notation** as $y', y'', \ldots, y^{(n)}$.

Example 1. Obtain the order and degree for the following differential equation.

$$\frac{dy}{dx} - \cos x = 0$$

Solution: The order and degree are evaluated as follows.

Given is,

$$\frac{dy}{dx} - \cos x = 0$$

Consequently, $y' - \cos x = 0$

Hence, the highest order of derivative $= 1$

Therefore,   Order $= 1$

Further since,   Degree $=$ Power of y'

Hence,   Degree $= 1$

The given differential equation is of Order 1 and Degree 1.

Example 2. Obtain the order and degree for the following differential equation.

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Solution: The order and degree are evaluated as follows.

Given is,

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Consequently, $xy\,y'' + x(y')^2 + y\,y' = 0$

Hence, the highest order of derivative $= 2$

Therefore,   Order $= 2$

Further since,   Degree $=$ Power of y''

Hence,   Degree $= 1$

The given differential equation is of Order 2 and Degree 1.

Example 3. Obtain the order and degree for the following differential equation.

$$y''' + y^2 + e^{y'} = 0$$

Solution: The the order and degree are evaluated as follows.

Given is,

$$y''' + y^2 + e^{y'} = 0$$

Hence, the highest order of derivative $= 3$

Therefore,   Order $= 3$

Further since,

        Degree $=$ Power of y'

Here y' is in $e^{y'}$ which is not a polynomial equation in derivatives.

Hence, Degree = Not Defined.

The given differential equation is of Order 3 and Degree is Not Defined.

Check Your Progress

1. What is a differential equation?
2. What is an ordinary differential equation?
3. What is an order of a differential equation?

10.3 SOLUTION OF DIFFERENTIAL EQUATIONS

A **Differential Equation** or **DE** comprises of **derivatives** or **differentials**. Some of the differential equations are solved using **integration**. A **differential** can be defined as a **derivative** such that where essentially *dy/dx* is not denoted or symbolized in the fraction form. Following are some frequently used examples of **differentials**.

- *dx* – Specifies that it is 'an infinitely small change in *x*'
- *d*θ – Specifies that it is 'an infinitely small change in θ'
- *dt* – Specifies that it is 'an infinitely small change in *t*'

General and Particular Solutions of a Differential Equation

The general and particular solutions of any differential equation is obtained as follows.

Definition. A **solution** of a **differential equation** is a **relation** between the **independent variable** and **dependent variable**, which is **free of derivatives** of **any order**, and which identically satisfies or justifies the differential equation.

Differential Equations Solutions

Assume that a common differential equation of *n*th order is of the form,

$$F\left[x, y, \frac{dy}{dx}, \ldots\ldots, \frac{d^n y}{dx^n}\right] = 0$$

Here '*F*' is considered as the **real function** of its (*n* + 2) **arguments** as,

$$x, \ y, \ dy/dx, \ \ldots\ldots\ldots, d^n y/d x^n$$

Consequently, we can define a **function** *f*(*x*) in an interval *x* ∈ **I** which includes *n*th **derivative** along with **all** the **derivatives** of **lower order** for all *x* ∈ **I** is termed as an '*explicit solution*' for the specified **differential equation** provided that the following condition is satisfied,

$$F[x,\ f(x),\ f'(x),\ f''(x), \ldots\ldots\ldots, f^{(n)}(x)] \ =\ 0 \text{ for all } x \in I$$

Additionally, a **relation** $g(x, y) = 0$ is termed as the **'implicit solution'** for the specified **differential equation** when as a minimum it expresses one **real function** **'f'** for the **variable 'x'** on the specified **interval 'I'** in such method that the function of the differential equation becomes an explicit solution on this interval as stated above.

10.3.1 General Solution of a Differential Equation

Definition. The **'General Solution'** of an **'nth order differential equation'** is the **unique method** that includes **'n' necessary (essential) arbitrary constants**.

For solving any differential equation of first order we can use the variables separable method. The arbitrary constant is necessarily or essentially defined when integration process is used for obtaining solution of differential equation. For example, after simplification or generalization the general solution of the first order differential equation essentially contains '1' arbitrary constant.

In the same way, after simplification or generalization the general solution of the second order differential equation essentially contains '2' arbitrary constants, and so on.

Fundamentally, geometrically the general solution of certain differential equation will represent an *n*-parameter family of curves. Figure 10.1 illustrates the family of curves having 1-parameter defining the general solution of the specified differential equation of the form,

$$dy/dx = 3x^2$$

This on simplification gives,

$$y = x^3 + c$$

Here '*c*' is an arbitrary constant.
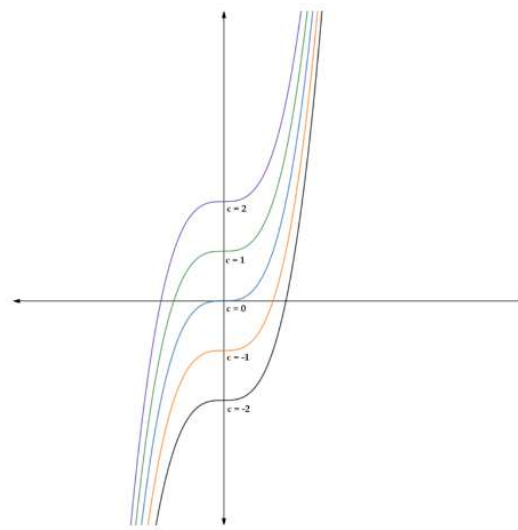
Fig. 10.1 *Parameter Family of Curves*

10.3.2 Particular Solution of a Differential Equation

Differential Equations

Definition. A **'Particular Solution'** of a specified **differential equation** is a **unique solution** which is obtained using the **'General Solution'**. For this we **assign specific** or **particular** values to the **arbitrary constants**. The necessary conditions to estimate the values of the arbitrary constants includes either the form of an initial value problem or of boundary conditions. We can also obtain the **Singular Solution** of a **Particular Solution** for the **specified differential equation** where we **will not use** the **General Solution**, i.e., by **not specifying** or **assigning** the **values** to the **arbitrary constants**.

NOTES

Example 4. Evaluate the following given function,

$$f(t) = c_1 e^t + c_2 e^{-3t} + \sin t$$

Then determine that whether the above equation is a general solution of the specified differential equation of the form,

$$\frac{d^2 F}{dt^2} + 2\frac{dF}{dt} - 3F = 2 \cos t - 4 \sin t$$

Also obtain the particular solution for the specified differential equation which must satisfy the initial value conditions where,

$$f(0) = 2 \text{ and } f'(0) = -5$$

Solution: The values are evaluated as follows for the specified differential equation.

Determination of the General Solution

The function $f(t)$ essentially satisfies the specified differential equation for obtaining the required solution. We will first define the derivatives of 'f' as follows:

$$f(t) = c_1 e^t + c_2 e^{-3t} + \sin t$$
$$f'(t) = c_1 e^t - 3c_2 e^{-3t} + \cos t$$
$$f''(t) = c_1 e^t + 9c_2 e^{-3t} - \sin t$$

For evaluating the result, on the left-hand side of the above given equation we will now use these defined values of 'F' to obtain the equation as follows,

$$\frac{d^2 F}{dt^2} + 2\frac{dF}{dt} - 3F = 2 \cos t - 4 \sin t$$

The equation on the left-hand side becomes,

$$(c_1 e^t + 9c_2 e^{-3t} - \sin t) + 2(c_1 e^t - 3c_2 e^{-3t} + \cos t) - 3(c_1 e^t + c_2 e^{-3t} + \sin t)$$

Now we cancel the like terms and on simplifying we obtain the following equation:

$$= 2 \cos t - 4 \sin t$$

This equation is similar to the equation given on the right-hand side. Consequently, the given function $f(t)$ is a solution for the specified differential equation.

Further,

Because the order of the given differential equation is of Order = 2

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Also the number of the Arbitrary Constants in the given Function $f(t) = 2$

Hence, the solution specified by the function $f(t)$ is certainly the General Solution of the given Differential Equation.

Determination of the Particular Solution

Given is the expression,

$$f(t) = c_1 e^t + c_2 e^{-3t} + \sin t$$

Considering the above expression at '$t = 0$' we obtain,

$$f(0) = c_1 + c_2 = 2 \qquad \qquad …(1)$$

Again consider the following expression,

$$f'(t) = c_1 e^t - 3c_2 e^{-3t} + \cos t$$

Then at '$t = 0$' we obtain,

$$f'(0) = c_1 - 3c_2 + 1 = -5 \qquad \qquad …(2)$$

We then solve the simultaneous linear Equation (1) and also the simultaneous linear Equation (2) to obtain the following values of c_1 and c_2.

$$c_1 = 0 \quad \text{and} \quad c_2 = 2$$

Therefore, the required or essential particular solution is given by the equation of the form,

$$f(t) = 2e^{-3t} + \sin t$$

Example 5. Evaluate the given equation for obtaining the particular solution.
$$y' = 5 \qquad \qquad \text{for} \quad x = 0 \quad \text{and} \quad y = 2$$

Solution: The values are evaluated as follows.

Given is, $\quad y' = 5$

Now we express in the form of differential equation as, $dy = 5\,dx$

On integrating both the sides of the equation we obtain, $\quad y = 5x + K$

Where K is a constant.

Given is the values of boundary conditions as,

$$x = 0 \quad \text{and} \quad y = 2$$

Therefore, $K = 2$

And hence, $y = 5x + 2$

The particular solution is $y = 5x + 2$.

10.4 VARIABLE SEPARABLE METHODS

Certain form of differential equations can only be solved using the particular method termed as the **'separation of variables'** or **'variables separable'**. This specified

method is possibly used only when the differential equation is expressed in the form,

$$A(x)\ dx + B(y)\ dy = 0$$

Here $A(x)$ is termed as the **function** of **only 'x'** while $B(y)$ is termed as the **function** of **only 'y'**.

Therefore, the differential equation is assumed or considered to be separable when we can express the equation by separating the variables. After expressing the given differential equation in the above format the equation is solved using the variable separable method through the process of integration for obtaining the desired general solution.

The variables separable method is specifically used for solving the differential equations of first order and first degree.

Following are three significant steps involved in the variable separable method.

Step 1. Put or arrange all the 'y' terms including the term 'dy' on one side of the equation and similarly put all the 'x' terms including the term 'dx' on the other side of the equation.

Step 2. Then start **integrating** one side of the equation with regard to 'y' and the other side of the equation with regard to 'x'. Always define or add in the equation '+ C' or '+ K', which is termed as the **constant of integration**.

Step 3. Simplify the equation to obtain the desired result.

Let us understand the concept with the help of the following example.

Given is the differential equation,

$$dy/dx = 5xy$$

We use the variable separable or separation of variables method for simplifying the given differential equation. For this we will first shift all the 'y' terms including the term 'dy' on one side of the equation, i.e., left-hand side as follows,

$$dy / y\ dx = 5x$$

Similarly, we will shift all the 'x' terms including the term 'dx' on the other side of the equation, i.e., right-hand side as follows,

$$dy / y = 5x\ dx$$

Example 6. Solve the given differential equation using the variable separable method.

$$2\ y\ dy\ =\ (x^2 + 1)\ dx$$

Solution: The differential equation is evaluated as follows.

This differential equation is expressed or stated in the variable separated form because the 'x' terms and 'y' terms are already separated. Hence, we only integrate for simplifying the given differential equation.

Given is, $\qquad 2\,y\,dy \;=\; (x^2 + 1)\,dx$

On integrating both sides we obtain,

$$\int 2\,y\,dy \;=\; \int (x^2 + 1)\,dx$$
$$y^2 \;=\; 1/3\,x^3 + x + C$$

Example 7. Solve the given differential equation using the variable separable method.

$$2\,\frac{dy}{dx} = \frac{y(x+1)}{x}$$

Solution: For solving the above given differential equation we will first separate the variables as follows.

Given is,

$$2\,\frac{dy}{dx} = \frac{y(x+1)}{x}$$

Applying the variable separable method, we obtain,

$$\frac{2\,dy}{y} = \frac{(x+1)\,dx}{x}$$

Consequently, we have,

$$\frac{2\,dy}{y} = \left(1 + \frac{1}{x}\right) dx$$

On integrating both sides, we have,

$$\int \frac{2\,dy}{y} = \int \left(1 + \frac{1}{x}\right) dx$$

Or, $\qquad 2\ln y = x + \ln x + K$

On expressing 'y' as explicit function of 'x', we have,

$$\ln y = \frac{x + \ln x + K}{2}$$

Hence, $\quad y = e^{(x + \ln x + K)/2}$

Figure 10.2 illustrates the solution graph when we take the characteristic constant value as, $K = 1$.
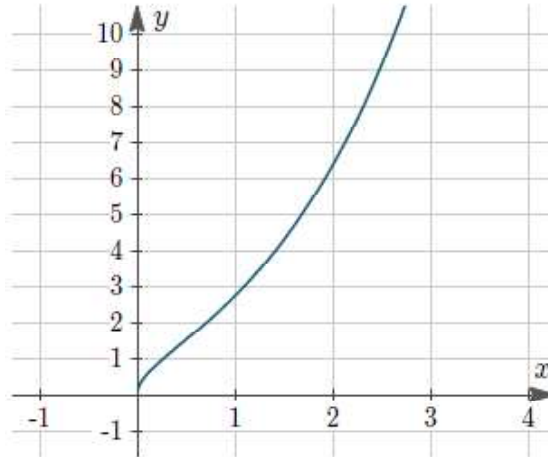
Fig. 10.2 *Characteristic Solution Graph for* $y = e^{(x + \ln x + K)/2}$

Example 8. Evaluate to obtain the particular solution for the given differential equation.

$$\frac{dy}{dx} + 2y = 6$$

Given is, $x = 0$ while $y = 1$.

Solution: The above differential equation is evaluated using the variable separable method as follows.

Given is,

$$\frac{dy}{dx} + 2y = 6$$

By separating the variables, we obtain,

$$\frac{dy}{dx} = 6 - 2y$$

$$dy = (6 - 2y) \, dx$$

$$\frac{dy}{6 - 2y} = dx$$

On integrating both sides we have,

$$\int \frac{dy}{6 - 2y} = \int dx$$

Subsequently,

$$\frac{-1}{2}\ln|6 - 2y| = x + K \qquad \dots(3)$$

Applying the given values in Equation (3) as $x = 0$ while $y = 1$, we have,

$$\frac{-1}{2}\ln|6 - 2(1)| = 0 + K$$

Consequently,

$$K = \frac{-1}{2}\ln|4|$$

We substitute this value of 'K' in Equation (3) and obtain,

$$\frac{-1}{2}\ln|6 - 2y| = x + \frac{-1}{2}\ln|4|$$

Simplifying we have,

$$\frac{-1}{2}[\ln|6 - 2y| - \ln 4] = x$$

$$\frac{\ln(6 - 2y)}{4} = -2x$$

When we take 'e' on both the sides then we have,

$$\frac{6 - 2y}{4} = e^{-2x}$$

$$6 - 2y = 4e^{-2x}$$

Hence, $y = 3 - 2e^{-2x}$

Therefore, $\dfrac{dy}{dx} = 4e^{-2x}$

Applying this value of 'y' to the left-hand side of the given differential equation we simplify as follows.

Given is,

$$\frac{dy}{dx} + 2y = 6$$

Applying the value in $y = 3 - 2e^{-2x}$ the above equation we have,

$$\frac{dy}{dx} + 2y$$

$$= 4e^{-2x} + 2\left(3 - 2e^{-2x}\right)$$

$$= 6$$

$$= \text{Right-Hand Side}$$

When specifically, $x = 0$, $y = 3 - 2e^0 = 1$.

Therefore, the particular solution is specified as $y = 3 - 2e^{-2x}$.

Figure 10.3 represents the solution graph for defining the curve that passes through $(0, 1)$.
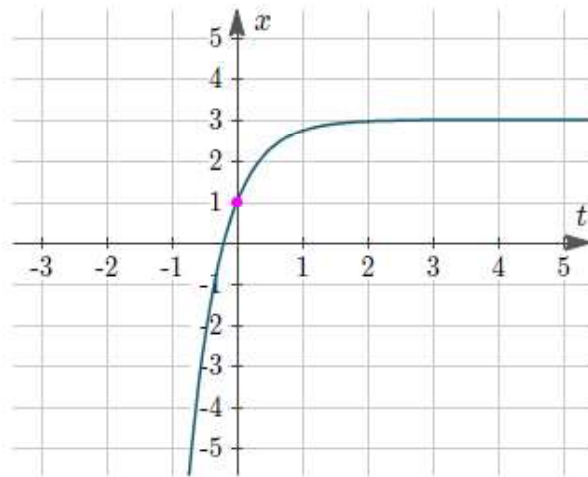
Fig. 10.3 *Solution Graph for* $y = 3 - 2e^{-2x}$

Check Your Progress

4. What is an explicit solution of a differential equation?

5. What is an implicit solution of a differential equation?

6. What form of differential equation is solvable by variable separation method?

10.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A differential equation is an equation for a function that relates the values of the function to the values of its derivatives.

2. An Ordinary Differential Equation (ODE) is an equation that contains an unknown function of one real or complex variable 'x', its derivatives and some specified functions of 'x'. Generally, the unknown function is characterized by a variable denoted by 'y', which consequently depends on 'x'. Hence, 'x' is termed as the independent variable of the equation.

3. The order of the differential equation is defined with regard to the order of the highest derivative included in the equation.

4. A function $f(x)$ in an interval $x \in$ I which includes nth derivative along with all the derivatives of lower order for all $x \in$ I is termed as an '*explicit solution*' for the specified differential equation provided that the following condition is satisfied,

$$F[x, \ f(x), \ f'(x), \ f''(x), \ldots\ldots\ldots, f^{(n)}(x)] \ = \ 0 \text{ for all } \ x \in \text{I}$$

5. A relation $g(x, y) = 0$ is termed as the '*implicit solution*' for a differential equation when as a minimum it expresses one real function 'f' for the variable 'x' on the specified interval 'I'.

6. Variables separable method is possibly used only when the differential equation is expressed in the form, $A(x) \ dx + B(y) \ dy = 0$. Here $A(x)$ is termed as the function of only 'x' while $B(y)$ is termed as the function of only 'y'.

10.6 SUMMARY

- A Differential Equation (DE) comprises of one or more expressions including derivatives of one dependent variable 'y' with reference to another independent variable 'x', such as

$$\frac{dy}{dx} = 2x$$

- An ordinary differential equation is specifically a differential equation that is defined for a single function. An ordinary differential equation comprises of ordinary derivatives. Therefore the ordinary differential equations manage the functions of a single variable and their derivatives.

- A Partial Differential Equation (PDE) is a differential equation of the form that contains unknown multivariable functions and their partial derivatives. Therefore the partial differential equations deal with functions of several variables.

- Differential equations are categorized with respect to order, since the order of a differential equation is defined with regard to the order of the highest order derivative that exists or occurs in the equation.

- The degree of any differential equation is defined with regard to the power of the highest order derivative that exists in the equation.

- Ordinary differential equations of nth-order has 'n' linearly independent solutions.

- A function $f(x)$ in an interval $x \in$ I which includes nth derivative along with all the derivatives of lower order for all $x \in$ I is termed as an '*explicit solution*' for the specified differential equation provided that the following condition is satisfied,

$$F[x,\ f(x),\ f'(x),\ f''(x),\ \ldots\ldots\ldots,f^{(n)}(x)]\ =\ 0 \text{ for all } x \in \text{I}$$

- A relation $g(x, y) = 0$ is termed as the '*implicit solution*' for a differential equation when as a minimum it expresses one real function 'f' for the variable 'x' on the specified interval 'I'.

- The variables separable method is specifically used for solving the differential equations of first order and first degree.

10.7 KEY WORDS

- **Differential equation:** An equation involving derivatives of a function or functions.

- **Derivative:** An expression representing the rate of change of a function with respect to an independent variable.

- **Integral:** An integral assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data.

10.8 SELF ASSESSEMNT QUESTIONS AND EXERCISES

Short Answer Questions

1. Obtain the order and degree for the following differential equation.

$$\frac{dy}{dx} - 2\sin x = 0$$

2. Obtain the order and degree for the following differential equation.

$$3y'' + 2y' - y = 0$$

3. Evaluate the given equation for obtaining the particular solution

   $y' = 10$ For $x = 2$ and $y = 3$.

4. Solve the given differential equation using the variable separable method.

   $y \, dy = (x + 5) \, dx$

Long Answer Questions

1. Describe variable separable method to solve differential equations with the help of an example.

2. Evaluate the given equation for obtaining the particular solution

   $y'' + 2y' = 5$, For $x = 0$ and $y = 1$.

3. Evaluate the given equation for obtaining the particular solution which must satisfy the initial value conditions where, $f(0) = 1$ and $f(0) = -5$

   $f(t) = c_1 e^t + c_2 e^{-6t} + 2\cos t.$

4. Solve the given differential equation using the variable separable method

   $10x\dfrac{dy}{dx} + y = (x+1) - (2x-1)$.

5. Evaluate to obtain the particular solution for the given differential equation

   $\dfrac{dy}{dx} + 6y = 12,$ Given is, $x = -1$ while $y = 1$.

10.9 FURTHER READINGS

Shah, Nita H. 2015. *Ordinary and Partial Differential Equations: Theory and Applications*. New Delhi: PHI Learning Pvt. Ltd.

Kapoor, N. M. 1997. *A Text Book of Differential Equations*. New Delhi: Pitambar Publishing.

Gupta, P.N. 2005. *Comprehensive Differential Equations: Paper II*. New Delhi: Laxmi Publications.

UNIT 11 HOMOGENEOUS EQUATIONS AND FIRST ORDER LINEAR EQUATIONS

Structure

11.0 INTRODUCTION

In this unit, you will learn to solve homogeneous differential equations and first order linear equations. In mathematics, an ordinary differential equation is a differential equation with one or more functions of one independent variable and its derivatives. Among ordinary differential equations, linear differential equations play a prominent role for several reasons. Most elementary and special functions in physics and applied mathematics are solutions of linear differential equations.

A differential equation can be homogeneous in either of two respects. A first order differential equation is said to be homogeneous if it may be written $f(x, y)$ $dy = g(x, y) dx$ where f and g are homogeneous functions of the same degree of x and y. In this case, the change of variable $y = ux$ leads to an equation of the form $\frac{dy}{x} = h(u)du$, which is easy to solve by integration of the two members.

A differential equation is homogeneous, if it is a homogeneous function of the unknown function and its derivatives. In the case of linear differential equations, this means that there are no constant terms. The solutions of any linear ordinary differential equation of any order may be put in deduced form by integration from the solution of the homogeneous equation obtained by removing the constant term.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Solve homogeneous equations by substitution method
- Solve first order linear equations

11.2 HOMOGENEOUS EQUATIONS IN '*x*' AND '*y*'

In mathematics, a **differential equation** is termed as a **homogeneous equation** when it is '*a homogeneous function of the unknown function and its derivatives*'.

Definition. A **differential equation** of the **first order** is termed as **homogeneous equation** when it is expressed as,

$$f(x, y)\, dy = g(x, y)\, dx$$

In this equation '*f*' and '*g*' are considered as the **homogeneous functions** of the similar or **equivalent degree** of '*x*' and '*y*'.

Definition. The **Ordinary Differential Equation (ODE)** of **first order** expressed in the following **differential form** is termed as a **homogeneous equation**.

$$P(x, y)\, dx + Q(x, y)\, dy = 0 \qquad \ldots(1)$$

In this Equation (1) both '*P*' and '*Q*' are the **homogeneous functions** of the similar or equivalent degree. Consistently it can be expressed in the form,

$$dy/dx = f(x, y)$$

Where the function $f(x, y)$ can be expressed as,

$$f(x, y) = g(y/x)$$

On substituting '*y* = *ux*' in the homogeneous equation will reduce the form of the homogeneous equation to an equation of the form which will include '*x*' as the independent variable and '*u*' as the new dependent variable where the variables can be separable.

Definition. The **Ordinary Differential Equation** or **ODE** of **First Order** when expressed in the below given form then it is termed as the **Homogeneous equation** type.

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

In this **Homogeneous equation**, both the functions $M(x, y)$ and $N(x, y)$ are considered as **Homogeneous Functions** of the similar **degree '*n*'**.

Now we multiply each variable with a **parameter 'λ'** and obtain the equations,

$$M(\lambda x, \lambda y) = \lambda^n M(x, y)$$

And, $\qquad N(\lambda x, \lambda y) = \lambda^n N(x, y)$

Therefore,

$$\frac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)} = \frac{M(x, y)}{N(x, y)}$$

Consequently, a **differential equation** of **first order** of the form,

$$dy/dx = f(x, y)$$

is termed as the **homogeneous equation** when the right-hand side of the equation for all '**t**' satisfies the condition,

$$f(tx, ty) = f(x, y)$$

Additionally, we can state that the right-hand side of the equation is termed as the **homogeneous function** of the **order '0'** with regard to the **variables '*x*'** and '*y*', represented as,

$$f(tx, ty) = t^0 f(x, y) = f(x, y)$$

We can also represent a homogeneous differential equation system as,

$$y' = f(x/y)$$

And in the differential system as,

$$P(x, y) \, dx + Q(x, y) \, dy = 0$$

In this equation, the function $P(x, y)$ and the function $Q(x, y)$ are the homogeneous functions having the similar or equivalent degree.

Definition of Homogeneous Function.

A function $P(x, y)$ is termed as a **homogeneous function** of **degree '*n*'** when the below given relationship is valid for all '$t > 0$':

$$P(tx, ty) = t^n P(x, y)$$

11.2.1 Solving Homogeneous Differential Equations

For solving homogeneous equation we can use the method of substitution, i.e., by substituting '$y = ux$' which will in turn result in the form or system of separable differential equation.

Consider the differential equation of the form,

$$(a_1 x + b_1 y + c_1) \, dx + (a_2 x + b_2 y + c_2) \, dy = 0$$

This differential equation system can be converted to the system of variable separable equation. For this we have to move or shift the origin of the coordinate system towards the point of intersection on the system of specified straight lines. The system of differential equation can only be transformed or converted into the system of variable separable equation when these straight lines exist parallel. We use the method change of variable given below to obtain the required solution,

$$z = ax + by$$

Example 1. Solve the following differential equation.

$$(2x + y)\, dx - xdy = 0$$

Solution: The given differential equation is solved as follows.

It is obvious that at *dx* and *dy* both the polynomials $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the first order. Hence, the given specific differential equation is also homogeneous.

Assume that $y = ux$.

Here '*u*' is defined as the new function that is dependent on '*x*'.

Subsequently we have,

$$dy = d(ux) = udx + xdu$$

We substitute these to the given differential equation as follows.

Given differential equation is,

$$(2x + y)\, dx - xdy = 0$$

On substituting we have,

$$(2x + ux)\, dx - x(udx + xdu) = 0$$

Therefore, on simplifying the above equation and then on dividing both the sides by '*x*' we obtain,

$$xdu = 2\, dx \quad \text{or} \quad du = 2\,(dx/x)$$

Integrating both sides of $du = 2\,(dx/x)$ we have,

$$\int du = 2 \int \frac{dx}{x} \quad \text{or} \quad u = 2\ln|x| + C.$$

Here '*C*' is the constant of integration.

Considering the previous variable '*y*' we can express,

$$y = ux = x\left(2\ln|x| + C\right)$$

Therefore, the given differential equation has the following two solutions:

$$y = x\left(2\ln|x| + C\right), \quad x = 0.$$

Example 2. Solve the following differential equation.

$$y' = \frac{y}{x} - \frac{x}{y}$$

Solution: The given homogeneous differential equation is solved as follows.

From the right-hand side of the equation it is evident that,

$$x \neq 0 \quad \text{and} \quad y \neq 0$$

On substituting $\quad y = ux,\ y' = u'x + u$

Consequently, we have,

$$u'x + u = \frac{u\not{x}}{\not{x}} - \frac{\not{x}}{u\not{x}}$$

$$\Rightarrow u'x + \not{u} = \not{u} - \frac{1}{u}$$

$$\Rightarrow \frac{du}{dx}x = -\frac{1}{u}$$

$$\Rightarrow u\,du = -\frac{dx}{x}$$

Now we integrate both the sides of this equation as follows,

$$\int u\,du = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{u^2}{2} = -\ln|x| + C$$

$$\Rightarrow u^2 = 2C - 2\ln|x|$$

Denote the constant $2C$ as only 'C' for solving the equation.

Therefore, we obtain,

$$u^2 = C - 2\ln|x| \quad \text{or} \quad u = \pm\sqrt{C - 2\ln|x|}.$$

Hence, we can write the general solution for the given differential equation as,

$$y = ux = \pm x\sqrt{C - 2\ln|x|}$$

Example 3. Solve the following differential equation.

$$(xy + y^2)\,y' = y^2$$

Solution: The given differential equation is solved as follows.

The given differential equation is of the form homogeneous equation.

It can also be represented as,

$$y' = \frac{y^2}{xy + y^2}$$

$$= \frac{\frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{y^2}{x^2}}$$

$$= \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x} + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$$

Now we make the substitution as, $y = ux$.

Subsequently, $y' = u'x + u$

We substitute these values of y and y' into the given differential equation to obtain,

$$(xux + u^2x^2)(u'x + u) = u^2x^2$$

$$\Rightarrow \quad ux^2(u + 1)(u'x + u) = u^2x^2$$

Simplify the equation by dividing both sides of the equation with ux^2.

The form '$x = 0$' cannot be the solution for the equation. We further find the solution for the given differential equation as,

$$u = 0 \quad \text{or} \quad y = 0$$

This can be one of the solutions for the given differential equation.

We obtain the result as,

$$(u + 1)(u'x + u) = u$$

$$\Rightarrow \quad u'x(u + 1) + u^2 + u = u$$

$$\Rightarrow \quad u'x(u + 1) = -u^2$$

$$\Rightarrow \quad \left(\frac{1}{u} + \frac{1}{u^2}\right) du = -\frac{dx}{x}$$

Now integrate both the sides of the equation for finding the general solution as follows.

$$\int \left(\frac{1}{u} + \frac{1}{u^2}\right) du = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \ln|u| - \frac{1}{u} = -\ln|x| + C \qquad \ldots(2)$$

We substitute '$u = y/x$' in the expression of Equation (2).

Then the Equation (2) becomes,

$$\ln\left|\frac{y}{x}\right| - \frac{1}{\frac{y}{x}} = -\ln|x| + C$$

$$\Rightarrow \quad \ln|y| - \ln|x| - \frac{x}{y} = -\ln|x| + C$$

$$\Rightarrow \quad y \ln|y| = Cy + x$$

The specified expression in the explicit inverse function $x(y)$ is denoted as,

$$x = y \ln|y| - Cy$$

Because, in the above expression 'C' is an arbitrary real number, hence before the constant 'C' the '$-$' sign can be replaced with the '$+$' sign to obtain the equation,

$$x = y \ln|y| + Cy$$

Therefore, the given or specified differential equation has the following solutions,

$$x = y \ln|y| + Cy, \quad y = 0$$

Check Your Progress

1. What is a homogeneous equation?

2. What is a homogeneous function?

11.3 FIRST ORDER LINEAR EQUATIONS

The **linear differential equations** of the **First Order** are the specific system of the differential equations for which the required **solution** is obtained with regard to the **variable coefficients**. Basically, nearly all the system of equations that are solved explicitly essentially involves the coefficients to be constant.

Definition. A **first order linear differential equation** is a **differential equation** of the form $y' + p(x)y = q(x)$.

Definition. The system of the **linear differential equation** of the **first order** can be expressed in the form,

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \ldots(3)$$

Where 'P' and 'Q' are termed as the **continuous functions** on a given or specified **interval**.

For example, the equation $xy' + y = 2x$ is a linear equation, since for '$x \neq 0$' can be expressed in the form,

$$y' + \frac{1}{x}y = 2 \qquad \ldots(4)$$

The above differential equation cannot be solved using the separable method because we cannot factor the y' expression by means of function of 'x' times a function of 'y'.

Consequently, this equation can be solved using the 'Product Rule' as,

$$x y' + y = (xy)'$$

Therefore,

$$(xy)' = 2x$$

On integrating the above equation on both the sides, we have,

$$xy = x^2 + C \quad \text{or} \quad y = x + (C/x)$$

Method 1. Essentially, all the **linear differential equations** of **first order** are solved by multiplying both the sides of the expression defined in Equation (3) using an appropriate function $I(x)$ termed as the *integrating factor*.

When Equation (3) is multiplied by the *integrating factor I(x)* then we obtain the *derivative* of the *product I(x) y* as,

$$I(x)(y' + P(x)y) = (I(x)y)' \qquad \qquad ...(5)$$

Subsequently, we can estimate the **function 'I'** and the Equation (3) becomes,

$$(I(x)y)' = I(x)Q(x) \qquad \qquad ...(6)$$

On integrating both sides of the Equation (4), we obtain,

$$I(x)y = \int I(x)Q(x)\,dx + C \qquad \qquad ...(7)$$

$$I(x)y = \int I(x)Q(x)\,dx + C$$

We obtain the following solution,

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right] \qquad \qquad ...(8)$$

For finding 'I' the Equation (5) is expanded and simplified by cancelling similar terms to obtain,

$$I(x)y' + I(x)P(x)y = (I(x)y)'$$
$$= '(x)y + I(x)y'$$
$$I(x)P(x) = I'(x)$$

This equation for **'I'** is termed as the separable differential equation and is solved as,

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln|I| = \int P(x)\,dx$$

$$I = Ae^{\int P(x)\,dx}$$

Here '$A = \pm e^c$'. Considering '$A = 1$' we obtain a *particular integrating factor* as,

$$I(x) = e^{\int P(x)\,dx}$$

For solving the linear differential equation of the form,

$$y' + P(x)y = Q(x) \qquad \qquad ...(9)$$

both the sides of the Equation (9) are multiplied by the integrating factor to integrate the equation.

$$I(x) = e^{\int P(x)\,dx}$$

Example 4. Solve the following differential equation.

$$\frac{dy}{dx} + 3x^2 y = 6x^2.$$

Solution: The above given differential equation is linear differential equation because it has the form of Equation (3) in which,

$$P(x) = 3x^2 \text{ and } Q(x) = 6x^2$$

Use the following integrating factor,

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

For solving the given linear differential equation, we multiply both the sides of the equation with e^{x^3} to obtain,

$$e^{x^3} \frac{dy}{dx} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

Then, $\dfrac{d}{dx}(e^{x^3}y) = 6x^2 e^{x^3}$

Integrate both the sides of the equation to obtain,

$$e^{x^3}y = \int 6x^2 e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

Example 5. Solve the following differential equation.

$$y' + 2xy = 1$$

Solution: The given differential equation '$y' + 2xy = 1$' has the standard linear equation form. We first find the integrating factor as,

$$e^{\int 2x\,dx} = e^{x^2}$$

On multiplying both the sides of the equation by the above integrating factor we obtain,

Hence, $e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$

Consequently, $e^{x^2}y = \int e^{x^2}\,dx + C$

Therefore, $y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$

Method 2. Considering the **first order linear differential equation** is a **differential equation** of the form,

$$y' + p(x)y = q(x) \qquad \text{…(10)}$$

For solving the **linear differential equation** we use the specific factor termed as an **integrating factor**. Fundamentally, an *integrating factor* can be defined as a specific **function** $f(x)$ which is used to multiply both sides of the Equation (10) as follows,

$$f(x)[y' + p(x)y] = f(x)q(x)$$
$$f(x)y' + f(x)p(x)y = f(x)q(x)$$

This function satisfies the equations of the form,

$$f(x)y' + f(x)p(x)y = f(x)y' + f'(x)y$$

$$\int \left[f(x)y' + f(x)p(x)y \right] dx = f(x)y + C$$

This specifies that the **function** 'f' must satisfy the following form of separable differential equation,

$$f'(x) = f(x)p(x)$$

On further simplifying we obtain,

$$\frac{f'(x)}{f(x)} dx = p(x) dx$$

$$\implies \int \frac{f'(x)}{f(x)} dx = \int p(x) dx$$

$$\implies \ln|f(x)| = \int p(x) dx$$

$$\implies f(x) = e^{\int p(x) dx} \qquad \text{…(11)}$$

Equation (11) represents the integrating factor which is essentially used for solving linear differential of first order.

Example 6. Solve the following differential equation.

$$y' + e^x y = e^x$$

Solution: The above given differential equation is solved as follows.

Because this equation is of the form,

$$p(x) = q(x) = e^x$$

Hence the integrating factor will be,

$$e^{\int e^x dx} = e^{e^x}$$

On multiplying both the sides of equation with e^{e^x} we obtain,

$$e^{e^x} y' + e^{e^x} e^x y = e^x e^{e^x}$$

$$\implies \left[e^{e^x} y \right]' = e^x e^{e^x}$$

On integrating both sides we obtain,

$$e^{e^x} y = e^{e^x} + C$$

$$\implies y = Ce^{-e^x} + 1$$

Example 7. Solve the following differential equation.

$$x \frac{dy}{dx} = x^2 + 3y$$

Solution: The given differential equation is solved as follows.

Since the given equation is of the form linear differential equation hence we will express this equation in the following standard form.

$$\frac{dy}{dx} - \frac{3}{x} y = x \qquad \qquad \dots(12)$$

Consequently, we define the integrating factor as,

$$I(x) = e^{\int P(x)dx}$$

Here $P(x)$ is considered as the coefficient of the 'y' term as,

$$I(x) = e^{\int \frac{-3}{x} dx}$$

$$= e^{-3 \ln x}$$

$$= (e^{\ln x})^{-3} = x^{-3}$$

On multiplying the standard form given in Equation (12), by $I(x) = x^{-3}$ we obtain,

$$x^{-3} \frac{dy}{dx} - \frac{3}{x^4} y = x^{-2}$$

$$\implies \frac{d[x^{-3} y]}{dx} = x^{-2}. \qquad \qquad \dots(13)$$

Now we integrate both the sides of the Equation (13) with respect to 'x', we obtain,

$$\int \frac{d[x^{-3}y]}{dx}\, dx = \int x^{-2}\, dx$$

$\Rightarrow \quad x^{-3}y = -x^{-1} + C$

Therefore, the solution is,

$$y = -x^2 + C\, x^3$$

Check Your Progress

3. Define first order linear differential equation.

4. Give an example of a linear equation which cannot be solved using the separable method.

11.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A differential equation of the first order is termed as homogeneous equation when it is expressed as, $f(x, y)\, dy = g(x, y)\, dx$. In this equation '$f$' and '$g$' are considered as the homogeneous functions of the similar or equivalent degree of 'x' and 'y'.

2. A function $P(x, y)$ is termed as a homogeneous function of degree 'n' when the below given relationship is valid for all '$t > 0$': $P(tx, ty) = t^n P(x, y)$.

3. A first order linear differential equation is a differential equation of the form $y' + p(x)y = q(x)$.

4. $xy' + y = 2x$ is a linear equation, since for '$x \neq 0$' can be expressed in the form,

$$y' + \frac{1}{x}y = 2$$

The above differential equation cannot be solved using the separable method because we cannot factor the y' expression by means of function of 'x' times a function of 'y'.

11.5 SUMMARY

- A differential equation of the first order is termed as homogeneous equation when it is expressed as, $f(x, y)\, dy = g(x, y)\, dx$. In this equation '$f$' and '$g$' are considered as the homogeneous functions of the similar or equivalent degree of 'x' and 'y'.

- The Ordinary Differential Equation (ODE) of first order expressed in the following differential form is termed as a homogeneous equation. In the below equation both '*P*' and '*Q*' are the homogeneous functions of the similar or equivalent degree.

$P(x, y)\, dx + Q(x, y)\, dy = 0$

- A function $P(x, y)$ is termed as a homogeneous function of degree '*n*' when the below given relationship is valid for all '$t > 0$': $P(tx, ty) = t^n P(x, y)$.

- A first order linear differential equation is a differential equation of the form $y' + p(x)y = q(x)$.

11.6 KEY WORDS

- **Homogenous function:** A homogeneous function is one with multiplicative scaling behavior: if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor.

- **Linear equation:** An equation between two variables that gives a straight line when plotted on a graph.

- **Differential equation:** An equation involving derivatives of a function or functions.

11.7 SELF ASSESSEMNT QUESTIONS AND EXERCISES

Short Answer Questions

1. Solve the following differential equation.

   $x + dx - xdy = 0$

2. Solve the following differential equation.

   $xdx - (x - 2y)dy = 0$

3. Solve the following differential equation.

   $(x + y)dy = x^2 dx$

Long Answer Questions

1. Solve the following differential equation.

   $$\frac{dy}{dx} - x^2 y = y^2 + 6x$$

2. Solve the following differential equation.

   $y' + xy = x - y$.

3. Solve the following differential equation.

   $2y' - e^x y = 5e^x + x$.

4. Solve the following differential equation.

$$2x\frac{dy}{dx} + x^2 = y^2$$

5. Solve the following differential equation.

   $(3xy - 2y^2) y' = y^2$

6. Discuss about homogeneous equation and first order linear equations with the help of examples.

11.8 FURTHER READINGS

Shah, Nita H. 2015. *Ordinary and Partial Differential Equations: Theory and Applications*. New Delhi: PHI Learning Pvt. Ltd.

Kapoor, N. M. 1997. *A Text Book of Differential Equations*. New Delhi: Pitambar Publishing.

Gupta, P.N. 2005. *Comprehensive Differential Equations: Paper II*. New Delhi: Laxmi Publications.

BLOCK - IV
VARIATIONS OF PARAMETERS, LAPLACE TRANSFORMATIONS AND STANDARD FORMS OF PARTIAL DIFFERENTIAL EQUATIONS

UNIT 12 LINEAR EQUATIONS OF ORDER 2 AND VARIATION OF PARAMETERS

Structure

12.0 Introduction
12.1 Objectives
12.2 Linear Equations of Order 2 with Constant and Variable Coefficients
12.3 Variation of Parameters
12.4 Answers to Check Your Progress Questions
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12.8 Further Readings

12.0 INTRODUCTION

In this unit, you will learn how to solve second order differential equations of a particular type, those that are linear and have constant coefficients. These equations are prominently used in the modelling of physical phenomena, for example, in the analysis of vibrating systems and the analysis of electrical circuits. In mathematics, variation of parameters, also known as variation of constants, is a general method to solve non-homogeneous linear ordinary differential equations.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe second order linear equations
- Solve linear differential equations by variation of parameters

12.2 LINEAR EQUATIONS OF ORDER 2 WITH CONSTANT AND VARIABLE COEFFICIENTS

Fundamentally, a second order differential equation is defined as an equation of the form which includes the unknown function '*y*', its derivatives as *y'* and *y''* and also the variable '*x*'.

The **second order linear ordinary differential equation** is expressed in the form,

$$y'' + a(x)y' + b(x)y = f(x) \qquad \qquad …(1)$$

In Equation (1), $a(x)$, $b(x)$ and $f(x)$ are the functions which are specifically defined on certain interval '*I*'.

Further, Equation (1) is termed as,

1. **Homogeneous** when $f(x) = 0$ for **ALL** $x \in I$.

2. **Non-Homogeneous** when $f(x) \neq 0$ for **SOME** $x \in I$.

Theorem 1. Existence and Uniqueness

Assume that $a(x)$, $b(x)$ and $f(x)$ are defined as the continuous functions on some interval '*I*'. Then for every $x_0 \in I$, $y_0 \in \mathbb{R}$, $z_0 \in \mathbb{R}$, there exists a unique solution '*y*' for Equation (1) such that,

$$y(x_0) = y_0 \qquad \qquad y'(x_0) = z_0$$

Second order linear homogeneous ordinary differential equation has the following given form when '$f(x) = 0$'.

$$y'' + a(x) y' + b(x) y = 0$$

Definition. The **linear differential equation** of **second order** or **order 2** has the form of equation as,

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x) \qquad \qquad …(2)$$

In Equation (2), the terms *P*, *Q*, *R* and *G* are termed as the **continuous functions**. In addition, when in Equation (2) if we have '$G(x) = 0$' for **all** '*x*' then the equation is termed as **homogeneous linear equation**. Therefore, the *second order homogeneous linear differential equation* has the form,

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0 \qquad \qquad …(3)$$

But if in Equation (2) we have '$G(x) \neq 0$' for **some** '*x*' then the equation is termed as **non-homogeneous linear equation**.

For solving the homogeneous linear equation if we assume that 'y_1' and 'y_2' are the *two solutions* of the *homogeneous linear equation* then the following

given *linear combination* is also considered as the solution for that homogeneous linear equation:

$$y = c_1 y_1 + c_2 y_2$$

Theorem 2. If $y_1(x)$ and $y_2(x)$ are **both solutions** of the **linear homogeneous** Equation (3) and, c_1 and c_2 are considered as the **constants**, then the following given function is also defined as a solution to Equation (3).

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Theorem 3. If y_1 and y_2 are **linearly independent solutions** of Equation (3) and $P(x)$ is never '**0**', then the general solution is given by the equation of the form,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad \qquad …(4)$$

Here, c_1 and c_2 are considered as the **arbitrary constants**.

To find the **particular solutions** for a **second order linear equation** we consider that the **coefficient functions P, Q and R** in Equation (3) are considered as the **constant functions**, i.e., when the differential equations are of the form,

$$ay'' + by' + cy = 0 \qquad \qquad ...(5)$$

Where a, b and c are constants and $a \neq 0$.

We consider that the **function 'y'** can be defined by means of constant times as y'' its second derivative, as y' another constant times and as y the third constant times which will be equal to '**0**'. As per the exponential function property $y = e^{rx}$, where 'r' being the constant, the derivative is a constant multiple of itself, i.e, $y' = re^{rx}$. Additionally, we have $y'' = r^2 e^{rx}$. Therefore, the Equation (5) has the solution as $y = e^{rx}$.

Definition: Second Order Linear Homogeneous Equations with Constant Coefficients

"A **second order ordinary differential equation** has the following general form for certain specified **function 'f'**."

$$y'' = f(t, y, y') \qquad \qquad …(6)$$

The Equation (6) is considered as **linear** when 'f' is **linear** in y and y' and is represented as,

$$y'' = g(t) - p(t) y' - q(t) y$$

If not then the equation will be defined as a **non-linear equation**.

Often the **second order linear equation** also has the form,

$$P(t) y'' + Q(t) y' + R(t) y = G(t) \qquad \qquad …(7)$$

When $G(t) = 0$ for all 't' then Equation (7) is termed as **homogeneous** and else **non-homogeneous**.

Characteristic Equation

For solving the second order differential equations with constant coefficients take the given equation of the form as expressed in Equation (5),

$$ay'' + by' + cy = 0$$

For Equation (5), as already discussed we assume that the solution has the form,

$$y = e^{rt}$$

We substitute this value of 'y' in the above differential equation and have the following equation,

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

Taking e^{rt} as common the equation is simplified as follows,

$$e^{rt} (ar^2 + br + c) = 0$$

Therefore, we obtain,

$$ar^2 + br + c = 0 \qquad \qquad \dots(8)$$

The Equation (8) is termed as the **characteristic equation** for the differential equation. The 'r' can be solved either by factoring or by quadratic formula.

General Solution

For obtaining the **general solution** we use the **quadratic formula** on the Equation (8), i.e., the **characteristic equation** of the form,

$$ar^2 + br + c = 0$$

We get r_1 and r_2 as two solutions.

Following are the three feasible or possible consequences (results):

- The roots r_1, r_2 are real and $r_1 \neq r_2$.
- The roots r_1, r_2 are real and $r_1 = r_2$.
- The roots r_1, r_2 are complex.

Then we can find 'r' as,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Assuming that roots r_1, r_2 are real and $r_1 \neq r_2$, then the general solution has the form,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Theorem 4. If the **characteristic equation** as stated in the Equation (8) has two **distinct real roots** r_1 and r_2 then the **general solution** for the Equation (5) is given by the equation of the form,

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Where c_1 and c_2 are arbitrary constants. If the **characteristic equation** as stated in the Equation (8) has only a **single** or repeated **real root** r then the **general solution** for the Equation (5) is given by the equation of the form,

$$y = (c_1 x + c_2) e^{r x}$$

Where c_1 and c_2 are arbitrary constants. If the **characteristic equation** as stated in the Equation (8) has **non-real** or **complex roots** $\alpha \pm \beta i$ then the **general solution** for the Equation (5) is given by the equation of the form,

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Where c_1 and c_2 are arbitrary constants.

Definition: Second Order Linear Differential Equations with Variable Coefficients

For the given **functions** $a_1, a_0, b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by the equation of the form,

$$y'' + a_1(t) y' + a_0(t) y = b(t) \qquad \ldots (9)$$

The Equation (9) is termed as the **second order linear differential equation** with **variable coefficients**. The Equation (9) is termed as **homogeneous iff** for all $t \in \mathbb{R}$ holds,

$$b(t) = 0$$

The Equation (9) is termed as **constant coefficients iff** a_1, a_0 and b are constants.

A **linear homogeneous second order equation** with **variable coefficients** is expressed as,

$$y'' + a_1(x) y' + a_2(x) y = 0 \qquad \ldots (10)$$

In the Equation (10), the terms $a_1(x)$ and $a_2(x)$ are defined as the **continuous functions** on the **interval [a, b]**.

Linear Independence of Functions

Definition. The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are termed as **linearly dependent** on the **interval [a, b]** when it includes **constants** as $\alpha_1, \alpha_2, \ldots, \alpha_n$, **not all '0'**, such that for all values of **'x'** from this interval, then the following identity holds.

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_n y_n(x) \equiv 0$$

When $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$, then the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are termed as **linearly independent** on the **interval [a, b]**.

Example 1. Solve the following equation for finding all solutions of '*y*'.

$$t\, y' = -2y + 4t^2$$

Solution: The given differential equation is solved as follows.

We will write the equation in the following form,

$$y' = -\frac{2}{t}\, y + 4t$$

$$\Rightarrow \quad y' + \frac{2}{t}\, y = 4$$

$$e^{f(t)}\, y' + \frac{2}{t}\, e^{f(t)}\, y = 4t\, e^{f(t)}.$$

$$f'(t) = \frac{2}{t}.$$

The function $\mu = e^{f(t)}$ is considered as an integrating factor.

$$f(t) = \int \frac{2}{t}\, dt = 2\,\ln(t) = \ln(t^2)$$

Consequently,

$$\mu(t) = e^{f(t)} = t^2$$

12.3 VARIATION OF PARAMETERS

The **variation of parameters** also sometimes termed as **variation of constants**, is a general method used for solving the **inhomogeneous** linear ordinary differential equations. Fundamentally, the **variation of parameters** method is a general method used for undetermined coefficients to find a particular solution for,

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = g(t) \qquad \qquad \text{…(11)}$$

But this method has two disadvantages. First, we need the complementary solution to solve the problem. Second, for solving the equation we have to work with a couple of integrals.

For deriving the formula for variation of parameters we have to acknowledge that the complementary solution to Equation (11) has the equation of the form,

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

The general solution for the homogeneous differential equation has,

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = 0$$

The complementary solution can be expressed for $y_1(t)$ and $y_2(t)$, a fundamental set of solutions. To obtain a pair of functions, $u_1(t)$ and $u_2(t)$ such that,

$$Y_P(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Definition. Variation of Parameter. For the differential equation of the form,

$$y'' + q(t)\, y' + r(t)\, y = g(t)$$

Consider that $y_1(t)$ and $y_2(t)$ are defined as the fundamental set of solutions for,

$$y'' + q(t)\, y' + r(t)\, y = 0$$

Consequently, the Wronskian and the particular solution for the non-homogeneous differential equation includes,

$$Y_P(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

Example 2. Solve the following differential equation for finding the general solution.

$$2y'' + 18y = 6\tan(3t)$$

Solution: The given differential equation is solved as follows.

Because the variation of parameters formula essentially involves a coefficient hence the differential equation will have the form,

$$y'' + 9y = 3\tan(3t)$$

Then for this differential equation the complementary solution has the form,

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore, we get the functions,

$$y_1(t) = \cos(3t) \qquad y_2(t) = \sin(3t)$$

Then for these two functions the Wronskian is defined as,

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

Consequently the particular solution is,

$$Y_P(t) = -\cos(3t) \int \frac{3\sin(3t)\tan(3t)}{3}\, dt + \sin(3t) \int \frac{3\cos(3t)\tan(3t)}{3}\, dt$$

$$= -\cos(3t) \int \frac{\sin^2(3t)}{\cos(3t)} \, dt + \sin(3t) \int \sin(3t) \, dt$$

$$= -\cos(3t) \int \frac{1 - \cos^2(3t)}{\cos(3t)} \, dt + \sin(3t) \int \sin(3t) \, dt$$

$$= -\cos(3t) \int \sec(3t) - \cos(3t) \, dt + \sin(3t) \int \sin(3t) \, dt$$

$$= -\frac{\cos(3t)}{3} \ln|\sec(3t) + \tan(3t)|$$

Similarly, the general solution is,

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) - \frac{\cos(3t)}{3} \ln|\sec(3t) + \tan(3t)|$$

CHECK YOUR PROGRESS

Check Your Progress

1. When is the equation $y'' + a(x)\, y' + b(x)\, y = f(x)$ homogeneous and non-homogeneous?

2. What is a second order linear differential equation?

3. If r_1 and r_2 as two solutions of the characteristic equation of the form, $ar^2 + br + c = 0$ then what are three possible consquences?

12.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. $y'' + a(x)\, y' + b(x)\, y = f(x)$ is homogeneous when $f(x) = 0$ for ALL $x \in I$ and non-Homogeneous when $f(x) \neq 0$ for SOME $x \in I$.

2. The linear differential equation of second order or order 2 has the form of equation as,

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x)$$

The terms P, Q, R and G are termed as the continuous functions.

3. (i) The roots r_1, r_2 are real and $r_1 \neq r_2$.

(ii) The roots r_1, r_2 are real and $r_1 = r_2$.

(iii) The roots r_1, r_2 are complex.

12.5 SUMMARY

- The second order linear ordinary differential equation is expressed in the form,

  $y'' + a(x) y' + b(x) y = f(x)$ where $a(x)$, $b(x)$ and $f(x)$ are the functions which are specifically defined on certain interval '*I*'.

- $y'' + a(x) y' + b(x) y = f(x)$ is homogeneous when $f(x) = 0$ for ALL $x \in I$ and non-Homogeneous when $f(x) \neq 0$ for SOME $x \in I$.

- The linear differential equation of second order or order 2 has the form of equation as,

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = G(x)$$

  The terms P, Q, R and G are termed as the continuous functions.

- A second order ordinary differential equation has the following general form for certain specified function '*f*', $y'' = f(t, y, y')$ is considered as linear when '*f*' is linear in y and y' and is represented as, $y'' = g(t) - p(t) y' - q(t) y$. If not then the equation will be defined as a non-linear equation.

- Characteristic equation of the form, $ar^2 + br + c = 0$. We get r_1 and r_2 as two solutions.

- The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are termed as linearly dependent on the interval $[a, b]$ when it includes constants as $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all '0', such that for all values of '*x*' from this interval, then the following identity holds. $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_n y_n(x) \equiv 0$ When $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$, then the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are termed as linearly independent on the interval $[a, b]$.

- The variation of parameters also sometimes termed as variation of constants, is a general method used for solving the inhomogeneous linear ordinary differential equations.

12.6 KEY WORDS

- **Homogenous function:** A homogeneous function is one with multiplicative scaling behavior: if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor.

- **Linear equation:** An equation between two variables that gives a straight line when plotted on a graph.

- **Differential equation:** An equation involving derivatives of a function or functions.

- **Continuous function:** A continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.

12.7 SELF ASSESSEMNT QUESTIONS AND EXERCISES

Short Answer Questions

1. Solve the following equation for finding all solutions of 'y'.

$$ty' = t^2 + y$$

2. Solve the following equation for finding all solutions of 'y'.

$$ty' = 3t - 2y$$

3. Solve the following equation for finding all solutions of 'y'.

$$ty' + y = t$$

Long Answer Questions

1. Solve the following differential equation by variation of parameters method for finding the general solution.

$$y'' - y^2 = tan\ t$$

2. Solve the following differential equation by variation of parameters method for finding the general solution.

$$5y'' = 10y - \cos(3t)$$

3. Solve the following differential equation by variation of parameters method for finding the general solution.

$$y'' - y = sin\ t$$

12.8 FURTHER READINGS

Shah, Nita H. 2015. *Ordinary and Partial Differential Equations: Theory and Applications*. New Delhi: PHI Learning Pvt. Ltd.

Kapoor, N. M. 1997. *A Text Book of Differential Equations*. New Delhi: Pitambar Publishing.

Gupta, P.N. 2005. *Comprehensive Differential Equations: Paper II*. New Delhi: Laxmi Publications.

UNIT 13   LAPLACE TRANSFORM

Structure

13.0 INTRODUCTION

In this unit, you will learn about Laplace transform and its properties. Further, you will learn to find inverse Laplace Tranform. The method of Laplace transforms is a system that relies on algebra to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential and integral equations. The Laplce Transform also has the advantage that it solves initial value problems directly without first finding a general solution. The ready tables of Laplace transforms make it easy to solve differential equations.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Define Laplace transform
- Find Inverse Laplace transform
- Solve differential equations using Laplace transform

13.2 LAPLACE TRANSFORM

The **Laplace transform** method is defined as a system that depend on algebra instead of calculus-based methods for solving linear differential equations. The Laplace transform method is specifically used for solving differential equations. In addition, this method is considered as an efficient and proficient alternative to variation of parameters method and uncertain coefficients. The Laplace transform method is specifically beneficial for input terms which are defined piecewise either periodic or impulsive.

Definition. Let $f(t)$ be defined for $t \geq 0$. The **Laplace transform** of $f(t)$ denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ is an **integral transform** given by the **Laplace integral**:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \, .$$

Provided that this **improper integral exists**, i.e., the integral must be **convergent**. Therefore the Laplace transform is a process that specifically transforms a function of 't', i.e., a function of *time domain* defined on $[0, \infty)$, to a function of 's', i.e., of *frequency domain*. Characteristically, $F(s)$ is the **Laplace transform** or basically just the **transform** of $f(t)$. The two **functions $f(t)$** and $F(s)$ are together termed as the **Laplace transform pair**.

When the **functions** of 't' are **continuous** on $[0, \infty)$, then the above defined **Laplace transformation** towards the **frequency domain** is '*one-to-one*', i.e., different types of the continuous functions will show different transformations. Additionally, the *kernel* for the Laplace transform is **unit less**, the e^{-st} in the **integrand**. Hence, the **unit** of 's' can be defined as **reciprocal** of 't', therefore 's' can be stated as a **variable** which will denote *complex frequency*.

Consider that $f(t) = 1$.

Then,

$$F(s) = \frac{1}{s}$$

And $s > 0$.

The Laplace transform is,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \frac{-1}{s} e^{-st} \Big|_0^\infty$$

The **integral** will be **divergent** for $s \leq 0$, though for $s > 0$ it will **converge** to,

$$\frac{-1}{s}\left(0 - e^0\right) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s)$$

Now consider that,

$$f(t) = e^{at},$$

Therefore,

$$F(s) = \frac{1}{s-a}$$

And $s > a$.

The Laplace transform is,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, e^{at}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$$

The **integral** will be **divergent** for $s \leq a$, though for $s > a$ it will **converge** to,

$$\frac{1}{a-s}\left(0 - e^0\right) = \frac{1}{a-s}(-1) = \frac{1}{s-a} = F(s)$$

Definition. A **function** $f(t)$ is termed as **piecewise continuous** if it has only finitely many discontinuities on any **interval [a, b]** and that both one-sided limits exist as '*t*' approaches each of those discontinuity from within the interval. Then '*f*' could have removable and/or jump discontinuities only and it cannot have any infinity discontinuity.

 Theorem 1. Assume that,

1. The '*f*' is considered as **piecewise continuous** on the interval $0 \leq t \leq A$ for any $A > 0$.

2. The $|f(t)| \leq K e^{at}$ when $t \geq M$, for any real constant '*a*' and some positive constants K and M. This specifies that 'f' is of exponential order.

Then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$.

This theorem states the sufficient condition for the existence of Laplace transforms and this cannot be a necessary condition. A function essentially may not satisfy the two conditions to have a Laplace transform.

Some Significant Properties of Laplace Transform

Following are some significant properties of Laplace transform:

1. $\mathcal{L}\{0\} = 0$

2. $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

3. $\mathcal{L}\{cf(t)\} = c\,\mathcal{L}\{f(t)\}$, for any constant c.

4. The derivative of Laplace transforms

$$\mathcal{L}\{(-t)f(t)\} = F'(s)$$

Or equivalently,

$$\mathcal{L}\{t f(t)\} = -F'(s)$$

Therefore,

$$\mathcal{L}\{t^2\} = -\left(\mathcal{L}\{t\}\right)'$$

$$= -\frac{d}{ds}\frac{1}{s^2}$$

$$= -\frac{-2}{s^3} = \frac{2}{s^3}$$

Properties 2 and 3 specify that **Laplace transform is linear**.

Fundamentally, the **derivatives** of **Laplace transform** satisfy,

$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s)$$

Or equivalently,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Table 13.1 defines the different types of Laplace transforms.

Table 13.1 *Types of Laplace Transforms*

$f(t) = 1,\ t \geq 0$	$F(s) = \frac{1}{s},\ s \geq 0$
$f(t) = t^n,\ t \geq 0$	$F(s) = \frac{n!}{s^{n+1}},\ s \geq 0$
$f(t) = e^{at},\ t \geq 0$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = \sin(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2+k^2}$
$f(t) = \cos(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2+k^2}$
$f(t) = \sinh(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2-k^2},\ s > |k|$
$f(t) = \cosh(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2-k^2},\ s > |k|$

Check Your Progress

1. How is Laplace transform defined?

2. When is a function termed as piecewise continuous?

3. What is the Laplace transform of $f(t) = e^{at}, t \geq 0$?

13.3 INVERSE LAPLACE TRANSFORM

Definition. When $\mathcal{L}\{F(t)\} = f(s)$ then $F(t)$ is said to be the **inverse Laplace transform** of $f(s)$. Hence we can express as,

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

Theorem 2. Linearity

The **inverse Laplace transform** is **linear** and is expressed as,

$$\mathcal{L}^{-1}\{a_1 f_1(s) + a_2 f_2(s)\} = a_1 \mathcal{L}^{-1}\{f_1(s)\} + a_2 \mathcal{L}^{-1}\{f_2(s)\}$$

Theorem 3.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$

Then $\mathcal{L}^{-1}\{f(s - a)\} = e^{at} F(t)$

Theorem 4.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$

Then, $\mathcal{L}^{-1}\{e^{-as} f(s - a)\} = \begin{cases} F(t - a), & t \geq a. \\ 0, & t < a. \end{cases}$

Definition. If we have the **transform** as $F(s)$ then for finding the **inverse Laplace transform** of $F(s)$ we use the given expression,

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Example 1. Find the inverse Laplace transform of the following;

$$F(s) = \frac{6}{s} - \frac{1}{s - 8} + \frac{4}{s - 3}$$

Solution. We find the inverse Laplace transform as follows.

Given is,

$$F(s) = 6\,\frac{1}{s} - \frac{1}{s-8} + 4\frac{1}{s-3}$$

Using the equation,

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

We have,

$$f(t) = 6(1) - e^{8t} + 4\left(e^{3t}\right)$$
$$= 6 - e^{8t} + 4e^{3t}$$

13.4 SOLVING DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

The Laplace transform can be used for solving the differential equations. Consider the following linear homogeneous ordinary differential equation of the form,

$$y'' - 5y' + 6y = 0 \quad y(0) = 2, y'(0) = 2$$

Consider that,

$$Y(s) = L[y(t)](s)$$

We will derive a new equation for $Y(s)$ for solving $y(t)$ by taking inverse transform.

Taking the Laplace transform on both the sides of the given differential equation, we obtain,

$$L[y'' - 5y' + y](s) = L[0](s)$$

The Laplace transform of the function '0' will be '0'.

Solving the left-hand side of the differential equation, we obtain,

$$L[y'' - 5y' + 6y](s) = L[y''](s) - 5L[y'](s) + 6L[y](s)$$

Using the Laplace transform formula for $L[y'']$ and $L[y']$ we have,

$$L[y'] = sL[y] - y(0) = sY(s) - 2$$

Applying $y(0) = 2$,

$$L[y''] = s^2 L[y] - sy(0) - y'(0) = s^2 Y(s) - 2s - 2$$

Consequently we obtain,

$$L[y''](s) - 5L[y'](s) + 6L[y](s) = s^2Y(s) - 2s - 2 - 5sY(s) + 10 + 6Y(s)$$

Therefore, the differential equation using Laplace transformed is,

$$(s^2 - 5s + 6)Y(s) - 2s - 2 + 10 = 0$$

This equation is considered as the linear algebraic equation for $Y(s)$.

We solve for $Y(s)$ to obtain,

$$Y(s) = \frac{2s - 8}{s^2 - 5s + 6}$$

Simplifying this expression with the help of partial fractions, we have,

$$Y(s) = \frac{4}{s - 2} + \frac{-2}{s - 3}$$

Using the inverse transforms we obtain,

$$L^{-1}[1/(s - 2)](t) = e^{2t} \quad L^{-1}[1/(s - 3)](t) = e^{3t}$$

The linearity method of the inverse transform is used to obtain,

$$y(t) = L^{-1}[\frac{4}{s - 2} + \frac{-2}{s - 3}](t) = 4e^{2t} - 2e^{3t}$$

Here 'L' is representing Laplace transform.

Example 2. Solve the given differential equation using Laplace transform method,

$$y' + 2y = 4t\,e^{-2t}, \qquad y(0) = -3$$

Solution: The differential equation is solved as follows.

On transforming using Laplace transform on both the sides of the equation,

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4t\,e^{-2t}\}$$

$$(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{4t\,e^{-2t}\}$$

$$= \frac{4}{(s + 2)^2}$$

Further simplify for finding $Y(s) = \mathcal{L}\{y\}$,

$$(s\mathcal{L}\{y\} - (-3)) + 2\mathcal{L}\{y\} = \frac{4}{(s + 2)^2}$$

$$(s + 2)\mathcal{L}\{y\} + 3 = \frac{4}{(s + 2)^2}$$

$$(s + 2)\,\mathcal{L}\{y\} = \frac{4}{(s+2)^2} - 3$$

$$\mathcal{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{s+2}$$

$$= \frac{4 - 3(s+2)^2}{(s+2)^3}$$

$$= \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

To obtain the inverse transform $y(t)$, we use partial fraction,

$$\mathcal{L}\{y\} = \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

$$= \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{s+2}$$

$$\frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b(s+2)}{(s+2)^3} + \frac{c(s+2)^2}{(s+2)^3}$$

$$= \frac{a + bs + 2b + cs^2 + 4cs + 4c}{(s+2)^3}$$

$$= \frac{cs^2 + (b+4c)s + (a+2b+4c)}{(s+2)^3}$$

Putting the values as,

$$-3 = c \qquad\qquad\qquad\qquad a = 4$$
$$-12 = b + 4c \qquad\qquad\quad b = 0$$
$$-8 = a + 2b + 4c \qquad\quad c = -3$$

$$\mathcal{L}\{y\} = \frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{4}{(s+2)^3} - \frac{3}{s+2}.$$

This expression resembles to the Laplace transform as,

$$2t^2 e^{-2t} - 3e^{-2t}.$$

Hence,

$$y(t) = 2t^2 e^{-2t} - 3e^{-2t}.$$

Check Your Progress

4. Define inverse Laplace transform.

5. Write the expression which shows that Laplace transform is linear.

13.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or is an integral transform given by the Laplace integral:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

2. A function $f(t)$ is termed as piecewise continuous if it has only finitely many discontinuities on any interval $[a, b]$ and that both one-sided limits exist as 't' approaches each of those discontinuity from within the interval. Then 'f' could have removable and/or jump discontinuities only and it cannot have any infinity discontinuity.

3. $F(s) = \frac{1}{s-a}, s > 0$.

4. When $\mathcal{L}\{F(t)\} = f(s)$ then $F(t)$ is said to be the inverse Laplace transform of $f(s)$. Hence we can express as,

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

5. $\mathcal{L}^{-1}\{a_1 f_1(s) \pm a_2 f_2(s)\} = a_1 \mathcal{L}^{-1}\{f_1(s)\} + a_2 \mathcal{L}^{-1}\{f_2(s)\}$

13.6 SUMMARY

- Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$ is an integral transform given by the Laplace integral:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

- $F(s)$ is the Laplace transform or basically just the transform of $f(t)$. The two functions $f(t)$ and $F(s)$ are together termed as the Laplace transform pair.

- A function $f(t)$ is termed as piecewise continuous if it has only finitely many discontinuities on any interval $[a, b]$ and that both one-sided limits exist as 't' approaches each of those discontinuity from within the interval. Then 'f' could have removable and/or jump discontinuities only and it cannot have any infinity discontinuity.

- When $\mathcal{L}\{F(t)\} = f(s)$ then $F(t)$ is said to be the inverse Laplace transform of $f(s)$. Hence we can express as,

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

- The inverse Laplace transform is linear and is expressed as,

$$\mathcal{L}^{-1}\{a_1 f_1(s) \pm a_2 f_2(s)\} = a_1 \mathcal{L}^{-1}\{f_1(s)\} + a_2 \mathcal{L}^{-1}\{f_2(s)\}$$

- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$

  Then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t)$

- If $\mathcal{L}^{-1}\{f(s)\} = F(t)$

  Then, $\mathcal{L}^{-1}\{e^{-as} f(s-a)\} = \begin{cases} F(t-a), & t \geq a \\ 0, & t < a \end{cases}$

- If we have the transform as $F(s)$ then for finding the inverse Laplace transform of $F(s)$ we use the given expression,

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

13.7 KEY WORDS

- **Improper integral:** An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.

- **Linear differential equation:** A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives.

- **Piecewise continuous:** A function is called piecewise continuous on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval and has a finite limit at the endpoints of each subinterval.

13.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write a short note on the properties of Laplace transform.

2. Find the inverse Laplace transform of the following;

$$F(S) = \frac{1}{S} + \frac{2}{S-1}$$

3. Find the inverse Laplace transform of the following;

$$F(S) = \frac{3S+5}{S}$$

4. Find the inverse Laplace transform of the following;

$$F(S) = \frac{S^2 - 1}{S+1}$$

5. Find the inverse Laplace transform of the following;

$$F(S) = \frac{S^2 + 2S - 1}{S+2}$$

Long-Answer Questions

1. Solve the given differential equation using Laplace transform method

$$y'' - 6y' + 15y = 2\sin 3t, y(0) = -1, y'(0) = -4$$

2. Solve the given differential equation using Laplace transform method

$$2y'' - 2y' + 3y = 3e^{3t}, y(0) = 1, y'(0) = 0$$

3. Solve the given differential equation using Laplace transform method

$$y'' - 3y' - 4y = 16, y(0) = -5, y'(0) = -4$$

4. Solve the given differential equation using Laplace transform method

$$y'' + e^{3t} = 3y, y(0) = -5$$

5. Solve the given differential equation using Laplace transform method

$$y'' + 2y = 3\cos t, y(0) = -1$$

13.9 FURTHER READINGS

Shah, Nita H. 2015. *Ordinary and Partial Differential Equations: Theory and Applications*. New Delhi: PHI Learning Pvt. Ltd.

Kapoor, N. M. 1997. *A Text Book of Differential Equations*. New Delhi: Pitambar Publishing.

Gupta, P.N. 2005. *Comprehensive Differential Equations: Paper II*. New Delhi: Laxmi Publications.

UNIT 14 PARTIAL DIFFERENTIAL EQUATIONS

Structure

14.0 INTRODUCTION

In this unit, you will study how to form a partial differential equation(PDE) and various methods of obtaining solutions of partial differential equation. In mathematics, a partial differential equation is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives. PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. In this unit, you will learn to form differential equations by eliminating arbitrary constants and variables. Further, this unit focuses on solving first order partial differential equations. Charpit's method and Lagrange's method for solving partial differential equations are discussed.

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss partial differential equations
- Form differential equations by eliminating arbitrary constants
- Form differential equations by eliminating arbitrary variables
- Solve first order partial differential equations

- Solve partial differential equations by using Charpit's method
- Describe Clairaut's form
- Solve partial differential equations by using Lagrange's method

NOTES

14.2 PARTIAL DIFFERENTIAL EQUATIONS

Typically, the equation which involves derivative (s) of the dependent variable with reference to the independent variable (s) is termed as the differential equation. Now let us understand about the **Partial Differential Equations** or **PDEs**.

Definition. The **differential equation** which involves the **derivatives** of the **dependent variable** with regard to only **one or single independent variable** is termed as an **Ordinary Differential Equation** or **ODE** while the **differential equation** which involves the **derivatives** with regard to **more than one independent variables** is termed as the **Partial Differential Equations** or the **PDEs**.

As already discussed in the previous units that the **'Order'** of the **differential equation** is the **order** of the **highest order derivative** taking place in the differential equation.

In addition, the **'Degree'** of a **differential equation** is expressed when it is a **polynomial equation** in its **derivatives**. Hence, the **'Degree'** of the **differential equation** for only the **positive integer** is stated as the **highest power** of the **highest order derivative** in it.

Definition. A **relation** between the **variables**, including the **dependent variable** and the **partial differential coefficients** of the **dependent variable** with the two or more **independent variables** is termed as a **Partial Differential Equation** or the **PDE**.

Following are some standard notations for partial differentiation coefficients:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + xy \qquad \ldots(1)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = u \qquad \ldots(3)$$

Therefore, since the order of a partial differential equation is expressed as the order of the highest order differential coefficient taking place in the equation while the degree of the partial differential equation is defined with regard to the degree of the highest order differential coefficient taking place in the equation, we can differentiate the above equations into different categories. For example, Equation

(1) is of first order first degree, Equation (2) is of second order first degree while Equation (3) is of second order third degree.

Further, when every single term of the equation holds either the dependent variable or one of its derivatives, then it is termed as the homogeneous form of equation else non-homogeneous form of equation. Hence, Equation (2) is **homogeneous** while Equation (1) is **non-homogeneous**.

The partial differential equation is stated as **linear** if the **differential co-efficients** that are taking place in the equation are of the first order only or alternatively if in each of the term, the differential coefficients are not in square or higher powers or their product and **non-linear** otherwise.

For example, the equation $x^2p + y^2q = z$ is considered as **linear** in **'z'** and is of **first order**.

Therefore, a **partial differential equation** is the unique form of equation which includes **one** or **more partial derivatives**. The **order** of the **highest derivative** is called the **order of the equation**. Fundamentally, a partial differential equation contains more than one independent variable.

Check Your Progress

1. What is the standard from of partial differential equation of first order first degree?

2. What is the standard from of partial differential equation of second order third degree?

3. What is the order of a differential equation?

14.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

The **partial differential equations** can be formed either by the **elimination of arbitrary constants** or by the **elimination of the arbitrary functions** from a relation with one dependent variable and the rest two or more independent variables.

Partial Differential Equations Obtained by the Elimination of Arbitrary Constants

When a *partial differential equation* is formed by **elimination** of **arbitrary constants** then we observe the following two conditions.

Condition 1. If the number of **arbitrary constants** are more than the number of **independent variables** in the given relations, then the *partial differential equation* obtained by **elimination** will be of **second** or **higher order**.

NOTES

Condition 2. If the number of **arbitrary constants** are equals the number of **independent variables** in the given relations, then the *partial differential equation* obtained by **elimination** will be of **order one**.

Partial Differential Equations Obtained by the Elimination of Arbitrary Functions

When the *partial differential equation* is formed by **elimination** of **arbitrary functions** then we observe the following condition.

Condition. When '*n*' is the number of arbitrary functions, then we may obtain several *partial differential equations*, but out of the obtained equations normally **one** with **two least order** is selected.

For example, consider the equation,

$$z = xf(y) + yg(x)$$

This equation includes **two arbitrary functions '*f*' and '*g*'**.

Thus, we have,

$$\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0 \qquad \qquad \ldots(4)$$

And, $xys = xp + yq - z$ $\qquad \qquad \ldots(5)$

The Equations (4) and (5) are the two partial differential equations which are obtained by **elimination of the arbitrary functions**. However, the Equation (5) is lower in order as compared to the Equation (4) so it is the desired partial differential equation.

Example 1. Form a partial differential equation by eliminating *a*, *b*, *c* from the relation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution: We obtain the solution as follows.

Evidently in the above given equation *a*, *b*, *c* are three arbitrary constants and *z* is a dependent variable which is dependent on *x* and *y*. The given relations can be expressed as follows:

$$f(x, y, z) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = 0 \qquad \qquad \ldots(6)$$

On differentiating Equation (6) partially with regard to *x* and *y* respectively, we have,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0, \qquad \qquad \text{Taking } \frac{\partial y}{\partial x} = 0$$

And, $\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z} \cdot \dfrac{\partial z}{\partial y} = 0$ \qquad Taking $\dfrac{\partial x}{\partial y} = 0$

Or, $\dfrac{2x}{a^2} + \dfrac{2z}{c^2} \cdot \dfrac{\partial z}{\partial x} = 0 \Rightarrow \quad c^2 x + a^2 z p = 0$ \hfill …(7)

And, $\dfrac{2y}{b^2} + \dfrac{2z}{c^2} \dfrac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad c^2 y + b^2 z q = 0$ \hfill …(8)

We again differentiate Equation (7) with respect to x, to obtain,

$$c^2 + a^2 \left(\dfrac{\partial z}{\partial x}\right)^2 + a^2 z \dfrac{\partial^2 z}{\partial x^2} = 0$$

Now substitute $\dfrac{c^2}{a^2} = -\dfrac{z}{x}\dfrac{\partial z}{\partial x}$ from Equation (7) in the above equation to obtain,

$$-\dfrac{z}{x}\dfrac{\partial z}{\partial x} + \left(\dfrac{\partial z}{\partial x}\right)^2 + z\dfrac{\partial^2 z}{\partial x^2} = 0$$

Or, $\quad xz \cdot \dfrac{\partial^2 z}{\partial x^2} + x\left(\dfrac{\partial z}{\partial x}\right)^z - z\dfrac{\partial z}{\partial x} = 0$ \hfill …(9)

Similarly, on differentiating Equation (8) partially with respect to 'y' and then substituting the value of $\dfrac{c^2}{b^2}$ from Equation (8) in the resultant equation, we obtain,

$$yz\dfrac{\partial^2 z}{\partial y^2} + y\left(\dfrac{\partial z}{\partial y}\right)^2 - z\dfrac{\partial z}{\partial y} = 0 \qquad …(10)$$

Consequently Equations (9) and (10) are the required 'partial differential equations' of first degree and second order.

Example 2. Form the partial differential equation by eliminating the arbitrary function from the following equation:

$$F(x + y + z, x^2 + y^2 + z^2) = 0$$

Solution: We form the partial differential equation as follows.

Let, $\quad F\big((x + y + z), (x^2 + y^2 + z^2)\big) = 0 \quad$ be $\quad F(u, v) = 0$ \hfill …(11)

Where, $u = (x + y + z) \quad$ and $\quad v = (x^2 + y^2 + z^2)$ \hfill …(12)

Evidently $F(u, v) = 0$ is an implicit function.

Therefore,

$$0 = F_x = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \ldots(i)$$

$$0 = F_y = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \ldots(ii)$$

$\ldots(13)$

However,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = (1 + p) \qquad \ldots(14)$$

$\left[\text{Since, } \dfrac{\partial y}{\partial x} = 0 = \dfrac{\partial x}{\partial y} \text{ as } x \text{ and } y \text{ are two independent variables}\right]$

And, $\quad \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial y} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial y} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial y} = (1 + q) \qquad \ldots(15)$

Similarly,

$$\frac{\partial v}{\partial x} = (2x + 2zp) \qquad \ldots(16)$$

$$\frac{\partial v}{\partial y} = (2y + 2zq) \qquad \ldots(17)$$

Consequently, we substitute the values of $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ in Equation (13), we obtain,

$$0 = \frac{\partial F}{\partial u}(1 + p) + \frac{\partial F}{\partial v}(2x + 2pz) \qquad \ldots(i)$$

$$0 = \frac{\partial F}{\partial u}(1 + q) + \frac{\partial F}{\partial u}(2y + 2qz) \qquad \ldots(ii)$$

$\ldots(18)$

Eliminating $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$ we obtain,

$$\begin{vmatrix} (1+p) & (2x + 2pz) \\ (1+q) & (2y + 2qz) \end{vmatrix} = 0$$

$$\Rightarrow \quad p(y - z) + q(z - x) = (x - y)$$

This is the desired partial differential equation.

14.4 FIRST ORDER PARTIAL ORDER EQUATIONS

A **differential equation** which includes only **first order partial differential coefficients '*p*'** and '*q*' is termed as **partial differential equation** of **first order**.

Additionally, if the degrees of '*p*' and '*q*' are unity only then it is termed as **linear partial differential equation** of **first order**. If each term of such an equation contains either the **dependent variable** or **one** of the **derivatives**, then the equation is considered to be **homogeneous** and **non-homogeneous** otherwise.

The **partial differential equation** of **first order** in **two independent variables** *x, y* and the **dependent variable** *z* can be written in the form,

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \qquad \ldots(19)$$

Also,

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

Equation (19) then takes the form,

$$f(x, y, z, p, q) = 0 \qquad \ldots(20)$$

Equation (20) is commonly used in various applications used in geometry and physics.

Example 3. Find all the functions $z(x, y)$ such that the tangent plane to the graph $z = z(x, y)$ at any arbitrary point $(x_0, y_0, z(x_0, y_0))$ passes through the origin characterized by the partial differential equation $xz_x + yz_y \, `` z = 0$.

Solution: We find the functions as follows.

The equation of the tangent plane to the graph at arbitrary point $(x_0, y_0, z(x_0, y_0))$ is,

$$z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0) - (z - z(x_0, y_0)) = 0.$$

This plane passes through the origin (0, 0, 0) and therefore, we must have,

$$-z_x(x_0, y_0)x_0 - z_y(x_0, y_0)y_0 + z(x_0, y_0) = 0 \qquad \ldots(21)$$

Equation (21) holds for all (x_0, y_0) in the domain of z, z and must satisfy,

$$xz_x + yz_y \, `` z = 0.$$

This is first order partial difference equation.

Check Your Progress

4. What will be the order of the partial differential equation obtained by elimination if the number of arbitrary constants are equals the number of independent variables in the given relations?

5. What is a linear partial differential equation of first order?

14.5 CHARPIT'S METHOD

In this section we will discuss the systems of first order partial differential equation and the Charpit's method for solving nonlinear partial differential equations. The Charpit's method is a general method for finding the complete integral of a nonlinear partial differential equation of first order of the form,

$$f(x, y, z, p, q) = 0 \qquad \qquad \text{...(22)}$$

Basic Notion: The basic notion of the Charpit's method is to introduce another partial differential equation of the first order of the form,

$$g(x, y, z, p, q, a) = 0 \qquad \qquad \text{...(23)}$$

It contains an **arbitrary constant '*a*'** and is solved as follows.

1. Equations (22) and (23) can be solved for '*p*' and '*q*' for obtaining,

$$p = p(x, y, z, a) \qquad q = q(x, y, z, a)$$

2. The equation,

$$dz = p(x, y, z, a)dx + q(x, y, z, a)dy \qquad \qquad \text{...(24)}$$

Equation (24) is integrable.

Then the solution is, $F(x, y, z, a, b) = 0$

Where the a, b are two arbitrary constants.

For solving the equation we introduce another partial difference equation g so that the equations f and g are compatible and then common solutions of f and g are determined using the Charpit's method.

Equations (22) and (23) are compatible if,

$$[f, g] \equiv \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0$$

On expanding we obtain the linear partial differential equation,

$$f_p\frac{\partial g}{\partial x} + f_q\frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z)\frac{\partial g}{\partial p} - (f_y + qf_z)\frac{\partial g}{\partial q} = 0. \qquad \text{...(25)}$$

On solving Equation (25) we determine g by finding the integrals of the following auxiliary equations:

$$\boxed{\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}} \qquad \text{...(26)}$$

These equations expressed in Equation (26) are termed as the Charpit's equations which are equivalent to the characteristics Equation (26).

Example 4. Find a complete integral of the given equation using Charpit's method,

$$p^2 x + q^2 y = z. \qquad \qquad \dots(27)$$

Solution: We find the complete integral as follows.

Step 1. Compute $f_x, f_y, f_z, f_p, f_q.$

Set, $f \equiv p^2 x + q^2 y - z = 0.$

Then,

$$f_x = p^2, \quad f_y = q^2, \quad f_z = -1, \quad f_p = 2px, \quad f_q = 2qy.$$

$$\implies pf_p + qf_q = 2p^2 x + 2q^2 y, \quad -(f_x + pf_z)$$

$$= -p^2 + p, \quad -(f_y + qf_z) = -q^2 + q.$$

Step 2. Write the Charpit's equations to find a solution $g(x, y, z, p, q, a)$.

The Charpit's equations are of the form,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\implies \frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2 x + q^2 y)} = \frac{dp}{-p^2 + p} = \frac{dq}{-q^2 + q}$$

Therefore,

$$\frac{p^2 dx + 2pxdp}{2p^3 x + 2p^2 x - 2p^3 x} = \frac{q^2 dy + 2qydq}{2q^3 y + 2q^2 y - 2q^3 y}$$

$$\implies \frac{p^2 dx + 2pxdp}{p^2 x} = \frac{q^2 dy + 2qydq}{q^2 y}$$

On integrating, we obtain,

$$\log(p^2 x) = \log(q^2 y) + \log a$$

$$\implies p^2 x = aq^2 y, \qquad \qquad \dots(28)$$

Where **a** is an arbitrary constant.

Step 3: Now we solve for **p** and **q** as follows.

Using Equations (27) and (28), we find that,

$$p^2 x + q^2 y = z, \quad p^2 x = aq^2 y$$

$$\implies (aq^2 y) + q^2 y = z \implies q^2 y(1 + a) = z$$

$$\implies \quad q^2 = \frac{z}{(1+a)y} \implies q = \left[\frac{z}{(1+a)y}\right]^{1/2}$$

And, $\quad p^2 = aq^2 \dfrac{y}{x} = a\dfrac{z}{(1+a)y}\dfrac{y}{x} = \dfrac{az}{(1+a)x}$

$$\implies \quad p = \left[\frac{az}{(1+a)x}\right]^{1/2}$$

Step 4: Expressing $dz = p(x, y, z, a)dx + q(x, y, z, a)dy$ and finding its solution.

$$dz = \left[\frac{az}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy$$

$$\implies \quad \left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy.$$

On integrating, we obtain,

$$[(1+a)z]^{1/2} = (ax)^{1/2} + (y)^{1/2} + b$$

This is the complete integral of the Equation (27).

14.6 CLAIRAUT'S FORM

Specifically in mathematical analysis, the **Clairaut's equation** or the **Clairaut equation** is a **differential equation** of the form,

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

Where f is **continuously differentiable**. It is a particular case of the Lagrange differential equation.

To solve Clairaut's equation, we differentiate with respect to x, which will yield,

$$\frac{dy}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2} + f'\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2};$$

Therefore,

$$\left[x + f'\left(\frac{dy}{dx}\right)\right]\frac{d^2y}{dx^2} = 0.$$

Hence, we have,

Either, $\dfrac{d^2 y}{dx^2} = 0$ …(29)

Or, $x + f'\left(\dfrac{dy}{dx}\right) = 0.$ …(30)

Considering the former case, i.e., Equation (29) we can state that $C = dy/dx$ for some **constant C**.

Now we substitute this into the **Clairaut's equation** to obtain the **family of straight line functions** given by,

$$y(x) = Cx + f(C).$$

This is considered as the **general solution** of Clairaut's equation.

Consider the latter case, i.e., Equation (30),

$$x + f'\left(\dfrac{dy}{dx}\right) = 0$$

This states only one or **singular solution y(x)**. The singular solution is generally represented using **parametric notation**, as **(x(p), y(p))**, where $p = dy/dx$.

By extension, a first order partial differential equation of the form,

$$u = xu_x + yu_y + f(u_x, u_y)$$

is also known as Clairaut's equation.

14.7 LAGRANGE'S MULTIPLIER METHOD

First order linear partial differential equation in its standard form is expressed as,

$$Pp + Qq = R \qquad …(31)$$

where **P, Q, R** are **functions** of x, y, z and is termed as the **Lagrange's Linear Equation**. This equation is obtained by **eliminating arbitrary function f** from,

$$f(u, v) = 0 \qquad …(32)$$

where u, v are **functions** of x, y, z.

Its solution depends on the solution of the equations,

$$\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} \qquad …(33)$$

Differentiating Equation (32) partially with regard to x and y respectively, we obtain,

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 0 \\[3mm] \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} = 0 \end{array}\right\}$$

Since, Equation (32) is an implicit relation.

More precisely,

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial u}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}p\right) + \dfrac{\partial f}{\partial v}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}p\right) = 0 \\[3mm] \dfrac{\partial f}{\partial u}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}q\right) + \dfrac{\partial f}{\partial v}\left(\dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z}q\right) = 0 \end{array}\right\} \qquad \ldots(34)$$

[Because $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial x}{\partial y}$ x and y being two independent variables.]

Then from Equation (34) on eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$, we obtain,

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}p \\[3mm] \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}q & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z}q \end{vmatrix} = 0 \qquad \ldots(35)$$

Implying,

$$\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right)p + \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right)q = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x} \qquad \ldots(36)$$

This is the similar as Equation (31) with the equations,

$$P = \left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right)$$

$$Q = \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right)$$

$$R = \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)$$

Now in order to find u and v, let $u = a$ and $v = b$, where a and b are two arbitrary constants, so that,

$$0 = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$0 = dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$ …(37)

From the above simultaneous equations, we obtain,

$$\frac{dx}{\dfrac{\partial u}{\partial y} \cdot \dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z} \cdot \dfrac{\partial v}{\partial y}} = \frac{dy}{\dfrac{\partial u}{\partial z} \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x} \dfrac{\partial v}{\partial z}} = \frac{dz}{\dfrac{\partial u}{\partial x} \dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y} \dfrac{\partial v}{\partial x}}$$

Or,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Solution of the above differential equations are **u = a** and **v = b**.

Therefore, the solution of Lagrange's Linear equation **Pp + Qq = R** is,

f(u, v) = 0   or   **f(a, b) = 0.**

Example 5. Solve the following Lagrange's Linear partial differential equation,

$$(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$$

Solution: We find the solution as follows.

Here the auxillary equations are,

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Or,

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - zx)} = \frac{dz}{(z^2 - xy)}$$

$$= \underbrace{\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)}}_{A} = \underbrace{\frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}}_{B} = \underbrace{\frac{dz - dx}{(z^2 - xy) - (x^2 - yz)}}_{C}$$

Taking expressions A and B we obtain,

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

Simplify to obtain,

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}$$

Or, $\dfrac{dx - dy}{(x - y)} = \dfrac{dy - dz}{(y - z)}$

Which is of the form,

$$\frac{f'(x)}{f(x)}$$

On integration, we obtain,

$$\log(x - y) = \log(y - z) + \log C_1 \quad \Rightarrow \quad C_1 = \left(\frac{x - y}{y - z}\right)$$

Similarly we solve for,

$$\frac{dy - dz}{(y - z)} = \frac{dz - dx}{(z - x)}$$

Or,

$$\log(y - z) = \log(z - x) + \log C_2 \quad \Rightarrow \quad C_2 = \left(\frac{y - z}{z - x}\right)$$

Hence, the desired solution of the given partial differential equation is,

$$f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0.$$

Check Your Progress

6. What is Clairaut's form of a differential equation?

7. Write Charpit's equations.

14.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u + xy$

2. $\left(\dfrac{\partial^2 u}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2 u}{\partial y^2}\right)^3 = u$

3. A partial differential equation is the unique form of equation which includes one or more partial derivatives. The order of the highest derivative is called the order of the equation.

4. If the number of arbitrary constants are equals the number of independent variables in the given relations, then the partial differential equation obtained by elimination will be of order one.

5. A differential equation which includes only first order partial differential coefficients 'p' and 'q' is termed as partial differential equation of first order. Additionally, if the degrees of 'p' and 'q' are unity only then it is termed as linear partial differential equation of first order.

6. $$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

7. $$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

14.9 SUMMARY

- A relation between the variables, including the dependent variable and the partial differential coefficients of the dependent variable with the two or more independent variables is termed as a Partial Differential Equation or the PDE.

- Following are some standard notations for partial differentiation coefficients:

$$, x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + xy, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = u$$

- A partial differential equation is the unique form of equation which includes one or more partial derivatives. The order of the highest derivative is called the order of the equation.

- The partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of the arbitrary functions from a relation with one dependent variable and the rest two or more independent variables.

- The partial differential equation of first order in two independent variables x, y and the dependent variable z can be written in the form,

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0.$$ Also, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. Then, $f(x, y, z, p, q) = 0.$

- Charpit's equations

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

- Clairaut's equation

$$y(x) = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

- First order linear partial differential equation in its standard form is expressed as,

  $Pp + Qq = R$

  where P, Q, R are functions of x, y, z and is termed as the Lagrange's Linear Equation

14.10 KEY WORDS

- **Order:** Order of the differential equation is the order of the highest order derivative taking place in the differential equation.

- **Degree:** Degree of the differential equation for only the positive integer is stated as the highest power of the highest order derivative in it.

- **Linear differential equation:** A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives.

- **Partial differential equation**: A partial differential equation is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives.

14.11 SELF ASSESSEMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What are the conditions observed when a partial differential equation is formed by elimination of arbitrary constants?

2. What are the conditions observed when a partial differential equation is formed by elimination of arbitrary functions?

3. What are some standard notations for partial differentiation coefficients?

4. Write a short note on Charpit's method.

5. Write a short note on Clairaut's form.

6. Write a short note on Lagrange's Multiplier method.

Long Answer Questions

1. Form a partial differential equation by eliminating *a*, *b* from the relation,

$$z = ax + by + a^2 + b^2.$$

2. Form a partial differential equation by eliminating *a*, *b*, *c* from the relation,

$$(1 + c)z = (a - 1)x + (b + 1)y.$$

3. Form the partial differential equation by eliminating the arbitrary function from the following equation,

$$z = xy + f(x^2 + y^2).$$

4. Form the partial differential equation by eliminating the arbitrary function from the following equation,

$$xyz = f(x + y + z).$$

5. Solve $\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 = \cos(2x - y)$ given that $z = 0$ when $y = 0$ and

$\frac{\partial z}{\partial y} = 0$ when $x = 0$.

6. Find a complete integral of the given equation using Charpit's method,

$$pxy + pq + qy = yz.$$

7. Solve the following Lagrange's linear partial differential equation,

$$p \tan x + q \tan y = \tan z.$$

14.12 FURTHER READINGS

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